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FOUNDATIONS

*Beth, E. W. **Sur la description de certains modèles d'un système formel.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 64–69. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

To illustrate the construction of a model (valuation of atomic formulae) by means of a limiting process, the author considers a system T_0 (set theory with types and a suitable axiom of infinity) and 'approximating' systems T_n (similar set theories with n individuals). The model M_n of T_n consists of the non-negative integers (nni) $<n$, sets of such integers, etc.; M_0 consists of all the nni, finite sets and their complements (w.r.t. all the nni), etc. M_0 may also be obtained as follows: Parameters (i) e_i, (ii) E_i, F_i, etc., are introduced for (i) individuals, (ii) sets, $\cdots$, and atomic formulae are formed with these parameters and the membership relation. If $a_i^{(n)}$ denotes the value of the parameter a_i in M_n, for $0 \leqq i < n$: $e_i^{(n)}$ is the integer i, for $i \geqq n$: $e_i^{(n)} = e_{n-1}^{(n)}$; for $0 \leqq j < 2^n$: $E_j^{(n)}$ is the set of exponents in the binary expansion of j and $F_j^{(n)} = E_{2^n-1-j}$, for $j \geqq 2^n$: $E_j^{(n)} = E_{2^n-1}^{(n)}$ and $F_j^{(n)} = E_0^{(n)}$. Then the truth value (t.v.) of each atomic formula is the same in all M_n for sufficiently large n and this limit is its t.v. in M_0. The numbering (naming) of the elements in M_n is crucial: e.g., the existence of infinite sets $F_j^{(0)}$ requires the second symbol $F_{2^n-1-j}^{(n)}$ for $E_j^{(n)}$, (for $j < 2^n$, $E_j^{(n)} = E_j^{(n+1)}$, but $F_j^{(n)} < F_j^{(n+1)}$). Even if the t.v. in M_n of a quantified formula tends to a limit this need not be its t.v. in M_0. The author points out that the axioms of choice and extensionality hold in M_0, but not all axioms of abstraction. Reviewer's note: The construction differs from the classical models of Gödel [see Hilbert and Bernays, Bd. II, Springer, Berlin, 1939, pp. 221–253] and Henkin [see Hasenjaeger, J. Symbolic Logic **18**, 42–48 (1953); these Rev. **14**, 1052] in the following respects: (i) both G and H introduce special parameters to eliminate existence symbols; (ii) their systems T_n are, in effect, the system T_0 itself with proofs restricted to lengths less than a suitable function of n; (iii) G has a whole set of models $M_n^{(i)}$ and proceeds to a lower limit; (iv) H introduces quantified formulae in the passage to the limit. *G. Kreisel* (Reading).

*von Wright, G. H. **A new system of modal logic.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. V, pp. 59–63. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

The author introduces into modal logic the symbol $M(p/q)$ which can be read "p is possible, given q". Then $\sim M(p/q)$ is read "p is impossible, given q". The only expressions dealt with are atomic M-expressions of the first order and their molecular complexes. Such expressions are called homogeneous M-expressions of the first order. Von Wright lays down the axioms:

A1. $M(t/t)$. A2. $\sim M(\sim t/t)$.
A3. $M(p/t) \rightarrow M(q/p) \vee M(\sim q/p)$.
A4. $M(p \& q/t) \leftrightarrow M(p/t) \& M(q/p)$.

He uses the rule "If the equivalence of two M-expressions of order zero can be proved in propositional logic, the expressions are intersubstitutible in M-expressions of order one". The letter t represents an arbitrary tautology of propositional logic and possibility is defined by $\Diamond p =_{df} M(p/t)$. The author proves two theorems in the system and then shows that the fragments of certain classical systems which contain only homogeneous modal expressions of the first order are proper subsystems of his system.

The concept of entailment is then discussed and von Wright conjectures that certain analogues of the paradoxes of strict implication cannot be derived in his system. He then discusses relationships between his system and the calculus of probability. The paper concludes with a brief discussion of modalities of higher order. *A. Rose.*

*Łukasiewicz, Jan. **A system of modal logic.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 82–87. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

This is a shortened version of another paper published by the author under the same title [J. Computing Systems **1**, 111–149 (1953); these Rev. **15**, 2]. *A. Rose.*

*Curry, Haskell B. **Les systèmes formels et les langues.** Les méthodes formelles en axiomatique. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 26, Paris, 1950, pp. 1–9; discussion, pp. 9–10. Centre National de la Recherche Scientifique, Paris, 1953.

Verf. erörtert die Beziehungen zwischen "formal system" und "Sprache". Ein formal system ist eine induktiv definierte Aussagenklasse. Es ist zu unterscheiden zwischen einer Darstellung und einer Interpretation eines formal system. Bei einer Darstellung werden die "obs" des formal system durch Objekte einer Klasse ersetzt, bei einer Interpretation werden die Prädikate so ersetzt, dass die Theoreme in wahre Aussagen übergehen. Zu jedem formal system gibt es eine Sprache und eine Metasprache. Die Metasprache ist wieder ein formal system, dessen Theoreme als Metatheoreme zu unterscheiden sind von den Epitheoremen. Jedes Metatheorem wird durch Interpretation zu einem Epitheorem, aber man erhält so nicht alle Epitheoreme. *P. Lorenzen* (Bonn).

Rose, Alan. The m-valued calculus of non-contradiction. J. Symbolic Logic **18**, 237–241 (1953).

The 2-valued calculus of non-contradiction of Dexter [Amer. J. Math. **65**, 171–178 (1943); these Rev. **4**, 126] has been extended to 3-valued logic by the author [Proc. London Math. Soc. (2) **54**, 184–200 (1952); these Rev. **14**, 123]. The methods used were too complicated to be capable of generalisation to m-valued logics. The object of the present paper is to give an alternative method of generalising Dexter's work to m-valued logics with one designated truth-value. The rule of procedure is generalised in the same way

as before, but the deductive completeness is proved by applying results of Rosser and Turquette [J. Symbolic Logic 14, 219–225 (1950); these Rev. 11, 709]. (From the author's summary.) *G. Kreisel* (Reading).

***Robinson, Abraham. Les rapports entre le calcul déductif et l'interprétation sémantique d'un système axiomatique.** Les méthodes formelles en axiomatique. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 36, Paris, 1950, pp. 35–51; discussion, pp. 51–52. Centre National de la Recherche Scientifique, Paris, 1953.

Verf. stellt an mehreren gut ausgewählten Beispielen seine Methode [On the metamathematics of algebra, North-Holland Publishing Co., Amsterdam, 1951; diese Rev. 13, 715] dar, aus dem Vollständigkeitstheorem des elementaren Prädikatenkalküls algebraische Sätze zu beweisen.

P. Lorenzen (Bonn).

Sampei, Yoemon. On the orthogonal expansion of the Boolean polynomial and its applications. II. Comment. Math. Univ. St. Paul. 1, 51–57 (1953).

Verf. hatte im Teil I [J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 113–125 (1950); diese Rev. 14, 4] für die von ihm definierten "formalen Quantoren" (D), (∀) und (∃) bewiesen, dass bei endlichem Individuenbereich R die Aussage $(\forall x)F_x$ eine geeignete Konjunktion von F_i $(i \in R)$ ist. Mit Hilfe des Auswahlprincips führt Verf. jetzt einen schwierigen Beweis auch für unendliches R durch. Die Möglichkeit einer "Degeneration", dass nämlich die auftretenden i nur eine echte Untermenge von R bilden, lässt sich, wie gezeigt wird, nicht durch eine Änderung der (impliziten) Definition der formalen Quantoren vermeiden. *P. Lorenzen* (Bonn).

Bar-Hillel, Yehoshua. A note on comparative inductive logic. British J. Philos. Sci. 3, 308–310 (1953).

Carnap, Rudolf. On the comparative concept of confirmation. British J. Philos. Sci. 3, 311–318 (1953).

Carnap, in his book "Logical foundations of probability" [Univ. of Chicago Press, 1950; these Rev. 12, 664] has defined the two relations $\mathfrak{ME}[h, e, h', e']$ (h is confirmed by e equal or more strongly than h' by e') and $\mathfrak{E}'[h, i, e]$ (h is confirmed by e and i together more strongly than by e alone). The first is supposed to be an explicatum for the comparative concept, the second one for the classificatory concept of confirmation. Carnap [op. cit., p. 466] shows by means of easy examples, that $\mathfrak{E}'$ is not an adequate explicatum for the classificatory concept of confirmation, in particular, that it is too narrow. In the first paper Bar-Hillel shows on the ground of Carnap's example for $\mathfrak{E}'$, that also $\mathfrak{ME}$ is not an adequate explicatum for the comparative concept of confirmation. Carnap, in the second paper, acknowledges the objections of Bar-Hillel and, instead of a new definition of $\mathfrak{ME}$, suggests an axiomatisation of this concept. *J. Łoś* (Toruń).

Behnke, Heinrich. Wandel im Aufbau der Mathematik. Math.-Phys. Semesterber. 3, 139–164 (1953).

Greenwood, Thomas. Valeur explicative des mathématiques. Rev. Trimest. Canad. 39, 172–182 (1953).

ALGEBRA

***Nasvytis, A. Die Gesetzmässigkeiten kombinatorischer Technik.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. viii+103 pp. DM 9.00.

This booklet considers combinatorial aspects of the selection of standards in industrial and commercial use. Some examples are: the size of the base in digital representation of numbers; weighing schemes (perfect partitions) and systems of fixed gauges; coding systems for telegraph, punched cards, and computing machinery. It is probably more interesting to systems engineers than to mathematicians.

For the most part, only simple combinatorial results are used, those enumerating the number of ways of distributing objects into boxes. What the author calls kinetic combinations and permutations are the number of ways of distributing unlike objects into like and unlike boxes (respectively), empty boxes allowed. For the last the number when n objects are distributed and s boxes each contain at least one object is $\Delta^s 0^n$, the differences of zero, an old result not noticed by the author. *J. Riordan.*

Roy, Purnendu Mohon. A note on the relation between bib and pbib designs. Science and Culture 19, 40–41 (1953).

The author states without proof four theorems, the last of which generalises a theorem of Shrikhande [Ann. Math. Statistics 23, 140 (1952)]. These theorems give methods for the construction of partially balanced incomplete block designs from balanced ones with certain specified properties. The construction consists in the deletion of certain varieties and certain blocks of the balanced design. All parameters of the resulting partially balanced designs are completely determined and, for three of the methods, every design with these parameters can be obtained by the author's method. In other words a balanced design can be reconstructed from every partially balanced one with the given parameters.

H. B. Mann (Columbus, Ohio).

***Ostrowski, A. M. Simultaneous systems of equations.** Simultaneous linear equations and the determination of eigenvalues, pp. 29–34. National Bureau of Standards Applied Mathematics Series, No. 29, U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

Let X denote an n-dimensional vector with components $x_1, \cdots, x_n$, and $f_\mu(X)=0$ a system of n equations for determining $x_1, \cdots, x_n$, with non-vanishing Jacobian. Let $\{I_{\nu,\mu}\}$ denote the matrix which is inverse to the Jacobian matrix, i.e., $I_{\nu,\mu}=\partial x_\nu/\partial f_\mu$. Let $\Delta_2(X)=(\sum_{\mu,\nu} I^2_{\nu,\mu})^{1/2}$. Let Y denote the vector with components $f_\mu(X)$. Let X_0 be a given value of X. The author shows that if there is a constant D and a neighborhood of X_0 specified by the condition $|X_0-X| \leqq D|Y_0|$, for which $\Delta_2(X) \leqq D$, then there is a solution of $f_\mu(X)=0$ in this neighborhood. The proof is based on the arc length of the curve $x_\nu=x_\nu(t)$ determined by the equations $f_\mu(X)-(1-t)f_\mu(X_0)=0$. The result can be generalized by replacing $\Delta_2(X)$ by $\Delta_p(X)=(\sum_{\mu,\nu} I^p_{\nu,\mu})^{1/p}$ and $|X_0-X|$ by a p norm and $|Y_0|$ by a q norm, for $p>1$. The author points out that a similar discussion can be based on the smallest and largest characteristic values of the Jacobian. A numerical example is given. Finally the basic argument of the theorem is applied to show the convergence of the Newton-Raphson method and to estimate the error.

F. J. Murray (New York, N. Y.).

*Narayan, Shanti. **A text book of matrices.** S. Chand & Co., Delhi, 1953. vii+289 pp.

This text begins with a brief account of groups, rings, and fields. A matrix is defined as a rectangular array of numbers and the algebra of matrices is developed from this point of view. In Chapter III rank is defined by using determinants, vectors are defined as ordered n-tuples and the main theorems on equivalence of matrices are proved over a general field. The next chapter treats the usual results on the congruent reduction of symmetric and hermitian matrices over the real and complex fields. The final chapter carries the similarity theory far enough to obtain necessary and sufficient conditions for a matrix to be similar to a diagonal matrix.

While the book clearly does not compete with recent books on the subject in which the concept of linear mapping is kept in the foreground, it will be useful to those who wish to keep the numerical components of matrices constantly in view. Attractive features include: definitions and theorems are stated with clarity and care; the format is attractive and the printing, paper and binding satisfactory; many numerical examples are solved in full detail and there is a large number of simple exercises. The book will be a useful addition to many libraries as a source of easy numerical exercises. *W. Givens* (New York, N. Y.).

Kotelyanskiĭ, D. M. On a property of sign-symmetric matrices. Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 163–167 (1953). (Russian)

Let the real $n\times n$ matrix B have the following property (sign-symmetry). The product of an arbitrary minor determinant (on rows $i_1, \cdots, i_k$ and columns $j_1, \cdots, j_k$) by the symmetrically situated minor determinant (on rows $j_1, \cdots, j_k$ and columns $i_1, \cdots, i_k$) is non-negative. Then B has certain properties enjoyed also by real symmetric matrices. I. Let D_r be a principal minor of dimension r=rank B, such that no principal minor of D_r is 0. Then the number of changes of sign in any "nested" sequence of principal minors $D_0=1, D_1, D_2, \cdots, D_r$ is an invariant of B. In this sequence, D_r involves indices $i_1, \cdots, i_r$, not necessarily in natural order, and D_k is the minor on rows and columns with indices $i_1, i_2, \cdots, i_k$. II. If the n leading principal minors $B(1), B(1, 2), \cdots$ are all positive, then all principal minors of B are positive. *J. L. Brenner.*

Wielandt, Helmut. Pairs of normal matrices with property L. J. Research Nat. Bur. Standards **51**, 89–90 (1953).

The set Z of all points z in the complex plane for which $A+zB$ is a normal matrix is either the whole plane, a straight line, a circle or at most two points and all six cases can occur. If B is normal, Z is the plane or a line. If Z is the plane, $AB=BA$. If for a fixed ordering of the eigenvalues α_k, β_k of A and B, the eigenvalues $\gamma_k(z)$ of $A+zB$ satisfy $\sum|\gamma_k|^2\geqq\sum|\alpha_k+z\beta_k|^2$ for values of z at the vertices of a polygon containing $z=0$ in its interior, then $AB=BA$.
W. Givens (New York, N. Y.).

Egerváry, E. On a property of the projector matrices and its application to the canonical representation of matrix functions. Acta Sci. Math. Szeged **15**, 1–6 (1953).

Theorem: If P is an $n\times n$ matrix of rank r such that $P^2=P$, then P can be written in the form BC^*, where B, C are $n\times r$, and $C^*B=I_r$ (identity). [If P is hermitian, then P can even be written BB^*, with $B^*B=I_r$.] In a different terminology, the author lets A be a linear transformation in a vector-space over an algebraically closed field, and lets the minimum polynomial $m(\lambda)$ of A satisfy $(m(\lambda), m'(\lambda))=1$. The components E_j of the canonical decomposition of the identity satisfy $E_j^2=E_j$. Set $E_j=B_jC_j^*$; then $C_j^*B_i=0$ $(i\neq j)$; $C_j^*B_j=I$. Thus the columns [rows] of the B_j [C_j] form a complete system of right- [left-] bi-orthogonal normal eigenvectors of A. This existence proof is constructive, and at the same time simpler than other discussions known to either the author or the reviewer. *J. L. Brenner.*

Ishaq, M. On four-point matrices. Ganita **4**, 61–78 (1953).

It is proved that a 4×4 matrix can be expressed as a sum $x=\sum_{p=0}^{15}c_pA_p$ (where c_p is the trace of xA_p) in terms of sixteen basic matrices A_p which appear in Dirac's electron theory. Multiplied by a specified power of the imaginary unit i $(i^2=-1)$, A_p is written ${}^r\alpha_{mn}$, where $m=0, 1, n=0, 1, 2, 3, 4$, and either $m=n=0$ or $0<m<n$, with r $(=1, 2, 3, 4)$ denoting the type of A_p. Each A or α is a four-point matrix, i.e., one having exactly one nonzero element in each row and each column, placed in the consecutive rows according to the column orders (1, 2, 3, 4), (2, 1, 4, 3), (3, 4, 1, 2), (4, 3, 2, 1) each of which four orders characterizes a type. The nonzero elements can only take the values ±1, $\pm i$, where i occurs four times in a matrix or else not at all; so there are 64 such matrices, 16 of each type r, or 8 pairs $\pm{}^r\alpha_{mn}$, where $m=0, 1, n=0, 1, 2, 3, 4$, and $m<n$ unless $m=n=0$. A nonzero m indicates a negative sign to the element at the mth row; likewise for n.

Properties of the A_p and of the ${}^r\alpha_{mn}$ are worked out (e.g., $A_p^2=I$), with special attention to multiplicative properties; thus the 64 matrices form a group, and its structure is investigated by dividing its multiplication table of 64 rows and columns into 8 equal panels. There is some difficulty in following the explanation which is rather too condensed; and the terms group and group character are not clearly used.
H. W. Turnbull (Millom).

*Permutti, Rodolfo. **Sulle equazioni algebriche a gruppo di Galois supersolubile.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 181–184. Casa Editrice Perrella, Roma, 1953.

Summary of results details of which have been published elsewhere [see the following review]. *W. Ledermann.*

Permutti, Rodolfo. Sulle equazioni algebriche a gruppo di Galois supersolubile. Ricerche Mat. **2**, 26–46 (1953).

Continuing his work on equations with supersoluble Galois groups [Giorn. Mat. Battaglini (4) **4**(80), 159–185 (1951); these Rev. **14**, 8] the author examines the Galois group, relative to a ground field K, of an equation $f(x)=0$ whose roots are all expressions of the form $t_1+t_2+\cdots+t_n$, where t_i ranges over all values obtained by eliminating $t_{i1}, t_{i2}, \cdots, t_{i,\nu_i-1}$ between the equations

$$t_i^{p_{i\nu_i}}=a_{i\nu_i}+t_{i,\nu_i-1},$$

$$t_{i,\nu_i-1}^{p_{i,\nu_i-1}}=a_{i,\nu_i-1}+t_{i,\nu_i-2},\ \cdots,\ t_{i2}^{p_{i2}}=a_{i2}+t_{i1},\quad t_{i1}^{p_{i1}}=a_{i1}.$$

Here p_{ij} $(i=1, 2, \cdots, n;\ j=1, 2, \cdots, \nu_i)$ are primes and the a_{ij} are assumed to be "general" elements of K. It is further supposed that K contains all the p_{ij}th roots of unity and that the characteristic of K is either 0, or else prime to all the p_{ij}.

It is shown that the Galois group of $f(x)=0$ is supersoluble if and only if, for each i, either $\alpha)$ $p_{i1}=p_{i2}=\cdots=p_{i\nu_i}$ or $\beta)$ $p_{i2}=p_{i3}=\cdots=p_{i\nu_i}$ and p_{i1} divides $p_{i2}-1$. Next, the author discusses equations whose roots are of the form

$$(a_0+a_1^{1/q_1}+a_2^{1/q_2}+\cdots+a_\nu^{1/q_\nu})^{1/p}$$

and finds necessary and, in some cases, sufficient conditions that the corresponding Galois group should be supersoluble. *W. Ledermann* (Manchester).

Raspanti, Matthew. A complex algebra for relay circuits. Elec. Engrg. **72**, 992–993 (1953).

Abstract Algebra

Fraïssé, Roland. Sur l'extension aux relations de quelques propriétés connues des ordres. C. R. Acad. Sci. Paris **237**, 508–510 (1953).

Two quasi-orderings $<$ and $\prec$ of the class of all $n$ary relations are defined as follows: $A<B$ means that A is isomorphic to the restriction $B|D$ of B to some subset D of its base, while $A \prec B$ means that $A|F<B$ whenever F is a finite subset of the base of A. Let γ_R be the class of all $A<R$ with finite base, and Γ_R the class of all $A \prec R$. A class of the form γ_R (Γ_R) is called a γ class (Γ class). The following results are announced. (1) The number of γ classes (Γ classes) of $n$ary relations is $\aleph_0$ for $n=1$ but $2^{\aleph_0}$ for $n \geqq 2$. (2) A class K of $n$ary relations with finite bases is a γ class if and only if ($\mathfrak{C}_1$) $A<B \in K$ implies $A \in K$, and ($\mathfrak{C}_2$) every two members of K have a common upper bound in K (with respect to $<$). A class K of $n$ary relations is a Γ class if and only if ($\mathfrak{C}_1$) and ($\mathfrak{C}_2$) hold, and also ($\mathfrak{C}_3$) every linearly ordered subset of K has an upper bound in K. (3) If K is a Γ class, then every subset of K has an upper bound in K. *B. Jónsson* (Providence, R. I.).

Fraïssé, Roland. Sur certaines relations qui généralisent l'ordre des nombres rationnels. C. R. Acad. Sci. Paris **237**, 540–542 (1953).

The terminology of the note reviewed above is used, and also the following definitions: R is said to be rich in K if R is a largest element of K (with respect to $<$). R is said to be homogeneous if, for every finite subset F of the base of R, every isomorphism of $R|F$ onto a restriction of R can be extended to an automorphism of R. The principal results are as follows. (1) If R is homogeneous and has a finite or countable base, then R is rich in the class of all $A \in \Gamma_R$ with finite or countable bases. (2) The following three conditions on a Γ class K are equivalent: (α) There exists a homogeneous relation R such that $K=\Gamma_R$. (β) There exists a homogeneous relation R with finite or countable base such that $K=\Gamma_R$. (γ) For any A, B in K and any set I contained in the bases of A and B, if $A|I=B|I$, then there exist $C \in K$ and isomorphisms φ and ψ of A and B onto restrictions of C such that $\varphi(x)=\psi(x)$ whenever $x \in I$. *B. Jónsson.*

Ellis, David. Notes on the foundations of lattice theory. II. Arch. Math. **4**, 257–260 (1953).

[For part I see Publ. Math. Debrecen **1**, 205–208 (1950); these Rev. **12**, 472.] Let L be a lattice. A mapping f of the cardinal product $L \otimes L$ onto an arbitrary set S is called a c-function on L if $f(a, b)=f(a \cap b, a \cup b)$ for all $a, b \in L$; examples are given. Theorems: L is distributive if and only if there is a c-function $f(a, b)$ on L which is one-to-one in b for fixed a. L is modular if and only if there is a c-function $f(a, b)$ on L such that $f(a, b)=f(a, c)$ and $b \neq c$ imply b and c are incomparable. The valuations on $L \otimes L$ of the form $f(x, y)=g(x)+g(y)$, where $g(x)$ is a valuation of L, are precisely the sum-symmetric real-valued c-functions on L. Symmetric differences in Boolean algebras are characterized. *P. M. Whitman* (Silver Spring, Md.).

Schützenberger, Marcel Paul. Le problème des mots dans les treillis modulaires libres. C. R. Acad. Sci. Paris **237**, 507–508 (1953).

The author claims a solution to the word problem for free modular lattices. From the very brief outline given, it is difficult to judge the validity of this claim. However, doubt is cast by the following counterexample to Lemma II:

$$X=(A+B)(C+AB), \quad X'=(A+B)C+AD,$$
$$Y=(A+B)(C+ABD)+AD$$

(modified from a suggestion of B. Jónsson). Since this problem is an outstanding unsolved one, further details will be awaited with great interest. Unfortunately, correspondence with the author indicates that full publication may be delayed by conditions beyond his control. Corrigenda: in definition of σ, for "C" read "A"; in Lemma III, for "$Z'\rho Y'\sigma X'''$" read "$Z'\rho^{-1}Y'\sigma X'''$"; in the Theorem, for "$\mathfrak{M}^*$" read "$\overline{\mathfrak{M}}^*$". *P. M. Whitman* (Silver Spring, Md.).

Utumi, Yuzo. On primal elements in a modular lattice. Kōdai Math. Sem. Rep. **1953**, 29–30 (1953).

The author applies his former study [same Rep. **1952**, 101–103; these Rev. **14**, 838] in order to extend some of C. W. Curtis's results on primal ideals [Amer. J. Math. **74**, 687–700 (1952); these Rev. **14**, 127] to modular lattices with maximum condition. Let L be such a lattice, Θ a set of congruences on L which are closed under intersection, and $a(\theta)$ (where $\theta \in \Theta$) the greatest element congruent to a. An element a is said to be primal with respect to Θ if $a(\theta_1 \cap \theta_2)=a$ implies $a(\theta_1)=a$ or $a(\theta_2)=a$. He shows that any primary or any meet-irreducible element is primal, and that any element is expressible as a normal (i.e., reduced and shortest) meet of a finite number of primal elements. If both $a_1 \cap a_2 \cap \cdots \cap a_n = b_1 \cap b_2 \cap \cdots \cap b_m$ are normal meets of primal components, then $m=n$ and after a suitable change of indices, $a_i(\theta)=a_i$ if and only if $b_i(\theta)=b_i$, $i=1, 2, \cdots, n$.

In case L is an integral modular lattice-ordered semigroup with maximum condition it is shown that primarity in the author's sense is the same as defined by Curtis [loc. cit.]. The results are finally applied to obtain the known facts concerning the special case of a modular lattice of finite length. *J. Levitzki* (Jerusalem).

Hashimoto, Junji. Ideal theory for lattices. Math. Japonicae **2**, 149–186 (1952).

The author introduces a topology on the family Ω_0 of all prime ideals of a lattice L by taking as a basis the family of all sets $F(x)=\{P \mid x \in P \in \Omega_0\}$ with $x \in L$. The first part of the paper is devoted to the study and applications of this topology. Since there is a natural 1-1 correspondence between the prime ideals of L and the prime ideals of the largest homomorph image of L which is distributive, the results obtained here follow rather trivially from known results [see M. H. Stone, Časopis Pěst. Mat. Fys. **67**, 1–25 (1937)]. The next topic concerns homomorphisms and ideals. Let $\mathfrak{L}$ be the lattice of all ideals of L and $\mathfrak{L}_N$ the sublattice consisting of all neutral elements of $\mathfrak{L}$, and for each $J \in \mathfrak{L}_N$ let θ_J be the smallest congruence relation over L whose kernel is J. The family $\Theta_N(L)=\{\theta_J \mid J \in \mathfrak{L}_N\}$ is a lattice of commuting equivalence relations. This fact gives rise to theorems analogous to the first and second isomorphism theorem and the Jordan-Hölder-Schreier theorem, where the equivalence relations involved are assumed to

belong to $\Theta_N(L)$. The decompositions of L are shown to correspond to complemented elements of $\Theta_N(L)$, and a solution is obtained to Birkhoff's Problem 73 [Lattice theory, rev. ed., Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948, p. 161; these Rev. 10, 673]. Finally the author studies maximal proper sublattices of a distributive lattice and shows that every proper sublattice can be extended to a maximal one. *B. Jónsson.*

Areškin, G. Ya. On congruence relations in distributive lattices with a zero element. Doklady Akad. Nauk SSSR (N.S.) **90**, 485–486 (1953). (Russian)

Let L denote a distributive lattice with 0, K a congruence relation on L, I a multiplicative ideal of L, and $\underline{K}(I)$ the smallest congruence relation on L with kernel I. The following results are stated without proof. Theorem 1: The greatest congruence relation $\bar{K}(I)$ with kernel I is the one with $x \equiv y$ if and only if $a \cap (x \cap y) \in I$ implies $a \cap (x \cup y) \in I$ for every $a \in L$. Theorem 2: If K has kernel I, then there is a unique congruence relation K^* on the factor lattice $L/\underline{K}(I)$ (whose zero is $\bar{0}$) such that $K = \underline{K}(I) \cdot K^*$, $\underline{K}(\bar{0}) \leqq K^* \leqq \bar{K}(\bar{0})$, and the mapping $K \to K^*$ is an isomorphism of the interval $[\underline{K}(I), \bar{K}(I)]$ onto $[\underline{K}(\bar{0}), \bar{K}(\bar{0})]$. Definition: A lattice is weakly complemented if for every pair $x \neq y$ in it, there exists z with $z \cap (x \cap y) = 0$, $z \cap (x \cup y) \neq 0$. Theorem 3: L is a generalized Boolean algebra if and only if for every I, $L/\underline{K}(I)$ is weakly complemented. Theorem 4: If K has kernel I, $L/K(I)$ is weakly complemented if and only if $K = \bar{K}(I)$. Theorem 5 [cf. G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, problem 73; these Rev. 10, 673]: The following are equivalent: (1) every congruence relation in L is uniquely determined by its kernel; (2) if $x, y \in L$ and $a \cap (x \cap y) \in I$ imply $a \cap (x \cup y) \in I$, $a \in L$, then there exists $a_0 \in L$ with $a_0 \cap (x \cap y) \in I$ and $a_0 \cup (x \cap y) \geqq x \cup y$; (3) for every I, $L/K(I)$ is weakly complemented; (4) L is a generalized Boolean algebra. [For closely related results, see the paper reviewed above.] *P. M. Whitman* (Silver Spring, Md.).

Hanai, Sitiro. On commutative T-closure operators. Kōdai Math. Sem. Rep. **1953**, 17–19 (1953).

An operator f which maps each element of a lattice L into an element a^f of L is called a T-closure operator if (1) $a \leqq a^f$, (2) $(a \cup b)^f = a^f \cup b^f$, (3) $(a^f)^f = a^f$, and (4) $0^f = 0$, where 0 is the zero element. Under a natural ordering the T-closure operators of a lattice form a complete lattice, whose properties were investigated by G. Birkhoff [Fund. Math. **26**, 156–166 (1936)]. In this paper the author defines the product fg of two such operators by $a^{fg} = (a^f)^g$, and considers cases when this product is commutative. He shows that $fg = gf$ implies $fg = f \cap g$, but not conversely, and also that if $fg \leqq gf$, then $fg = gf$ if and only if $fg = f \cap g$. A necessary and sufficient condition for commutativity of f and g is $fgf = fg$ and $gfg = gf$. It follows that f and g are commutative if and only if fg and gf are both T-closure operators.

Other necessary and sufficient conditions for commutativity are found. Finally it is shown that if G is a maximal subset of T-closure operators such that each pair of elements of G commute, then G is a complete sublattice of the complete lattice F of all T-closure operators, with preservation of infinite as well as finite meets. *O. Frink.*

Aumann, Georg. Über Hüllen- und Kernbildungen auf Verbänden. J. Reine Angew. Math. **191**, 50–53 (1953).

The author extends to lattices the notions definable in topological spaces through the combined use of kernel and envelope operations (such as interior and closure). Let A be a complete lattice, and F and G be sublattices which are complete, corresponding to the sublattices of closed and open sets in topology. For h an element of A, define the envelope h^s to be the infimum of all $f \in F$ such that $h \leqq f$, and the kernel h^i to be the supremum of all $g \in G$ such that $g \leqq h$. It follows that $h^{xx} = h^x$ for $x = s, i, (si), (is)$.

A set of elements (g, f) of G and F is called an associated pair if $g^s = f$ and $f^i = g$. It is shown that g is the lower element of such a pair if and only if $g^{si} = g$, and f likewise is an upper element if and only if $f^{is} = f$. The pairs (h^{is^i}, h^{is}) and (h^{si}, h^{si^s}) are always associated. The set G of all h with $h^{si} = h$ forms a complete sublattice of G consisting of all lower elements of associated pairs. For a topological space, G is the Boolean algebra of all regular open sets. The corresponding set F of all upper elements specializes to the Boolean algebra of all regular closed sets.

Other applications of the general theory are given. Let A be the lattice of all real-valued functions, and G and F the lattices of all lower and upper semicontinuous real-valued functions defined over a topological space. A function f is called minimally discontinuous if (f^i, f^s) is an associated pair. It is shown that if f is a minimally discontinuous function defined over a complete metric space, there is no function which coincides with f at all points of continuity and has fewer points of discontinuity than f.

Another application is to the set of all mappings of a partially ordered set into a complete lattice. Here F and G consist of all isotone and anti-isotone mappings. The special case of mappings of Boolean algebras into the real numbers is related by the author to Carathéodory's abstract approach to the theory of integration. [illegible] *O. Frink.*

Nôno, Takayuki. On the mutual connectedness of elements in lattices. J. Sci. Hiroshima Univ. Ser. A. **17**, 1–9 (1953).

Considering a binary relation γ on a lattice L, the author selects several postulates about γ which enable him to develop theorems analogous to those about addition and decomposition of connected sets in topology. The relationship between γ and mappings of L is studied in the case L is complete, and in particular a complete Boolean algebra. Thus if L is complete, there exists a mapping $x \to x^a \in L$ such that $x \gamma y$ is given by $y \leqq x^a$, if and only if $y \leqq \bigvee_{z \gamma u} u$ implies $x \gamma y$; and then x^a is the latter union. *P. M. Whitman.*

Copeland, Arthur H., Sr., and Harary, Frank. The extension of an arbitrary Boolean algebra to an implicative Boolean algebra. Proc. Amer. Math. Soc. **4**, 751–758 (1953).

The first author has previously shown [Math. Z. **53**, 285–290 (1950); these Rev. **12**, 721] that in an implicative Boolean algebra B every non-zero principal ideal has the same cardinal as B. It follows that many Boolean algebras (e.g., atomistic ones) cannot be made into implicative Boolean algebras. The purpose of the paper under review is to show that every Boolean algebra can be extended to an implicative Boolean algebra. There appears to be an error in the proof: Let S be the set of all positive integers, with their natural well-ordering, and let B be the set-field generated by the sets

$$a = \{1, 3, 5, 7, \cdots\}, \quad b = \{2, 4, 8, 12, 16, \cdots\},$$
$$c = \{6, 10, 14, 18, \cdots\}.$$

Then the set $b \cdot T_{b \vee c}(a)$ consists of the number 2 alone,

while according to the third paragraph in the proof of Theorem 1 this set should be infinite. *B. Jónsson.*

Foster, Alfred L. Generalized "Boolean" theory of universal algebras. I. Subdirect sums and normal representation theorem. Math. Z. **58**, 306–336 (1953).

A binary algebra is a system $\mathfrak{U}=(U, \times, \theta, \cdots)$ with elements $0, 1 \in U$ such that $0\times\xi=\xi\times 0=0$ and $1\times\xi=\xi\times 1=\xi$ for $\xi \in U$. A $\mathfrak{U}$ function (strict $\mathfrak{U}$ function) is a function expressible in terms of the operations and elements of (the operations of) $\mathfrak{U}$. If there is a permutation $\hat{}$ of U with $0\hat{}=1$ and $1\hat{}=0$, such that both $\hat{}$ and its inverse are $\mathfrak{U}$ functions, then $\mathfrak{U}$ is called an f algebra. If every function on U^h to U is a $\mathfrak{U}$ function (strict $\mathfrak{U}$ function), then $\mathfrak{U}$ is said to be functionally complete (strictly functionally complete). If $\mathfrak{U}$ is a finite f algebra and J a Boolean algebra, then the J extension $\mathfrak{U}=(\mathbf{U}, \times, \cdots, \theta, \cdots)$ of $\mathfrak{U}$ is defined as follows: $\mathbf{U}$ is the set of all partitions $[\cdots, a_\mu, \cdots]$ $(\mu \in U)$ of J, and if $\xi=[\cdots, x_\mu, \cdots]$, $\eta=[\cdots, y_\mu, \cdots]\cdots$, then

$$[\theta(\xi, \eta, \cdots)]_\mu = \sum_{\theta(\alpha,\beta,\cdots)=\mu} (x_\alpha y_\beta \cdots).$$

The main results are: 1. $\mathfrak{U}$ is isomorphic to a subalgebra of $\mathfrak{U}$, and every identity of $\mathfrak{U}$ extends to $\mathfrak{U}$. 2. Every $\mathfrak{U}$ function can be expressed in a normal form in terms of its values on U. 3. The various J extensions of $\mathfrak{U}$ can be characterized, up to isomorphism, as certain subalgebras of powers $\mathfrak{U}^N$ of $\mathfrak{U}$. 4. If $\mathfrak{U}$ is functionally complete and $\mathfrak{B}$ is a superalgebra of $\mathfrak{U}$ such that every identity of $\mathfrak{U}$ extends to $\mathfrak{B}$, then $\mathfrak{B}$ is isomorphic to some J extension of $\mathfrak{U}$. These results are applied to systems such as finite fields and Post algebras. Generalizations are also considered where $\mathfrak{U}$ is infinite. In this case J must be complete and atomistic. (The author assumes only completeness, but he also uses atomisticity; see, e.g., the proof of Lemma 4 on page 323.) The J extension of $\mathfrak{U}$ is then isomorphic to $\mathfrak{U}^N$ where N is the cardinal of the set of all atoms in J. *B. Jónsson* (Providence, R. I.).

Foster, Alfred L. Generalized "Boolean" theory of universal algebras. II. Identities and subdirect sums of functionally complete algebras. Math. Z. **59**, 191–199 (1953).

This is a continuation of the paper reviewed above. Among the results are the following. 1. If $\mathfrak{U}$ and $\mathfrak{A}$ are algebras of the same species with two or more elements, if $\mathfrak{U}$ is finite and strictly functionally complete, and if every identity of $\mathfrak{U}$ is also an identity of $\mathfrak{A}$, then $\mathfrak{A}$ is isomorphic to a subalgebra of a power of $\mathfrak{U}$. 2. The set of all identities which hold in a finite strictly functionally complete algebra $\mathfrak{U}$ is maximal in the sense that every larger set of identities fails in every algebra with two or more elements.

B. Jónsson (Providence, R. I.).

Riguet, Jacques. Sur les algèbres extérieures et les algèbres de Clifford en tant que produits croisés et sur la notion de complexe simplicial. C. R. Acad. Sci. Paris **237**, 638–639 (1953).

On peut définir les algèbres extérieures et les algèbres de Clifford comme des produits croisés d'un corps par un anneau booléen. Il y a équivalence entre le concept de système de facteurs de Grassmann et le concept de complexe simplicial. On pourrait généraliser en supposant que l'anneau booléen est remplacé par un p-anneau.

Author's summary.

Dubreil-Jacotin, Marie-Louise. Propriétés algébriques des transformations de Reynolds. C. R. Acad. Sci. Paris **236**, 1950–1951 (1953).

Following J. Kampé de Fériet [Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 165–180 (1949), p. 171; these Rev. **11**, 336], the author [C. R. Acad. Sci. Paris **236**, 1136–1138 (1953); these Rev. **14**, 839] has already discussed Reynolds operators $T: f\to Tf$ on the ring $R(E, K)$ of all functions on a set E, which assume only a finite number of distinct values in an ordered field K. The present paper extends this discussion, proving for example that the values assumed by Tf do not depend on the values assumed by f on "T-annulling" subsets X of E (i.e., subsets such that $TC_X=0$, where C_X denotes the characteristic function of X). *G. Birkhoff.*

Greger, Karl. Multilinear forms and elementary divisors in a principal ideal ring. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **23**, no. 17, 7 pp. (1953).

Let R be a principal ideal ring, f a multilinear form with coefficients in R, d the greatest common divisor of the coefficients. The author gives a new proof of a theorem attributed to Frobenius and Smith: it is possible to specialize the variables in f so that f takes the value d. The proof uses only the following property of principal ideal rings: given a and b with $(a, b)=d$, there exist x and y with $(x, y)=(y, d)=1$ and $ax+by=d$. An equivalent condition was used in a similar way by the reviewer [Trans. Amer. Math. Soc. **66**, 464–491 (1949); these Rev. **11**, 155]. The author further proves a refined result due to M. Riesz, asserting that any multiple of d admits a primitive representation by f. In the second part of the paper application is made to the theory of finitely generated R-modules; in particular, there is a characterization by M. Riesz of the elementary divisors.

I. Kaplansky (Chicago, Ill.).

Fujiwara, Izuru. On the evaluation of the operator function $\log(e^y e^z)$. Progress Theoret. Physics **9**, 676 (1953).

It is a well known fact that, for noncommuting operators y and z, the operator function $x=\log(e^y e^z)$ can be expressed in terms of y and z by an infinite succession of linear combinations and formation of commutators. [For early proofs, see H. F. Baker, Proc. London Math. Soc. (2) **3**, 24–47 (1905); F. Hausdorff, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. **58**, 19–48 (1906).] The author expresses z as $e^{-u}ye^u$, where u is determined by the explicit expression of its commutators with y and z. There is a similar solution of form $x=e^{-v}ze^v$. *L. Van Hove* (Princeton, N. J.).

Kuroš, A. G. Radicals of rings and algebras. Mat. Sbornik N.S. **33(75)**, 13–26 (1953). (Russian)

Let there be given a class of rings, S-rings. An S-ideal of a ring is a two-sided ideal which is an S-ring. The S-radical of a ring is an S-ideal containing all other S-ideals of the ring. The author considers an S-radical to be defined for all rings if: (1) S-rings are preserved under homomorphisms; (2) every ring has an S-radical; (3) a ring modulo its S-radical is S-semi-simple, i.e., contains no nonzero S-ideals. The S-radical $\leqq T$-radical if every ring which is T-semi-simple is also S-semi-simple. It is then shown that if M is any class of rings which contains a nonzero homomorphic image of every two-sided ideal of a ring in M, it is possible to define a certain S-radical, the upper radical of M. Under it all rings in M are semi-simple and it is $\geqq$ any S-radical under which all rings of M are semi-simple. The author then shows that for any class of rings N it is possible to define a

certain S-radical, the lower radical of N, which turns all rings of N into radical rings and which is $\leq$ any S-radical under which all the rings of N are radical rings. Furthermore, if C_1 and C_2 is any splitting of all simple rings into two nonvoid nonintersecting classes there exist S-radicals under which C_1 consists of semi-simple and C_2 of radical rings. For any such S-radical, upper radical of $C_1 \geqq S$-radical $\geqq$ lower radical of C_2. Finally, splittings C_1 and C_2 are given which correspond to the various radicals defined hitherto. The question whether these radicals coincide with the upper radical of C_1 or the lower radical of C_2 is partially solved.

All the results carry over directly to any algebraic system which admits a definition of homomorphism and kernel of homomorphism with the usual properties. There is some overlapping with a recent paper by Amitsur [Amer. J. Math. 74, 774–786 (1952); these Rev. 14, 347], but the author states that his results were obtained independently.

A. Rosenberg (Evanston, Ill.).

Nagao, Hirosi, and Nakayama, Tadasi. On the structure of (M_0)- and (M_u)-modules. Math. Z. 59, 164–170 (1953).

Let A be an algebra of finite dimension with 1-element. A left module $\mathfrak{m}$ of A is said to be an (M_0)-module if every A-space homomorphic to $\mathfrak{m}$ contains a direct summand isomorphic to $\mathfrak{m}$. Dually, $\mathfrak{m}$ is said to be an (M_u)-module if it is a direct summand of every module of which it is a subspace. It is not required that $\mathfrak{m}$ be finite-dimensional, nor that the 1-element of A be unit operator on $\mathfrak{m}$, but it is assumed that the unit scalar is unit operator on $\mathfrak{m}$. It is proved that $\mathfrak{m}$ is an M_0-module if and only if $1\mathfrak{m}$ is a direct sum of modules isomorphic to left ideal components of A and that $\mathfrak{m}$ is an M_u-module if and only if $1\mathfrak{m}$ is a direct sum of modules whose duals are isomorphic to right ideal components of A. (A component left ideal of A is understood to be a direct summand of A.) *R. M. Thrall.*

*Kochendörffer, Rudolf. **Erweiterungen von Gruppen und Körpern.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 455–459. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

An account of this investigation has already appeared in Math. Nachr. 2, 245–250 (1949); these Rev. 11, 314.

R. Brauer (Cambridge, Mass.).

Tomber, Marvin L. Lie algebras of type F. Proc. Amer. Math. Soc. 4, 759–768 (1953).

Over an algebraically closed field of characteristic zero the 52-dimensional simple Lie algebra $\mathfrak{F}_4$ is the derivation algebra of the exceptional simple 27-dimensional Jordan algebra $\mathfrak{M}_3^8$ [Chevalley and Schafer, Proc. Nat. Acad. Sci. U. S. A. 36, 137–141 (1950); these Rev. 11, 577]. Over any field of characteristic zero a Lie algebra is of type $\mathfrak{F}$ if upon extension of the base field it becomes $\mathfrak{F}_4$, and a Jordan algebra is of type $\mathfrak{M}$ if it becomes $\mathfrak{M}_3^8$. In this paper the relation between the algebras of these two classes is shown to be the same as that between Cayley algebras and Lie algebras of type $\mathfrak{G}$ [N. Jacobson, Duke Math. J. 5, 775–783 (1939); these Rev. 1, 100]. The facts are as follows: (1) The algebras of type $\mathfrak{F}$ are exactly the derivation algebras of the central simple algebras of type $\mathfrak{M}$. (2) If $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are such Jordan algebras then every isomorphism of the derivation algebra $\mathfrak{D}(\mathfrak{J}_1)$ onto $\mathfrak{D}(\mathfrak{J}_2)$ is induced by an isomorphism of $\mathfrak{J}_1$ onto $\mathfrak{J}_2$. *W. G. Lister* (Providence, R. I.).

Campbell, H. E. Concerning Cartan's criterion. Proc. Amer. Math. Soc. 4, 515–518 (1953).

The problem considered here is the relation between the radical $\mathfrak{R}(\mathfrak{L})$ of a Lie algebra $\mathfrak{L}$ over a field Φ of characteristic zero and the radical of the form $(x, y) = \mathrm{tr}(\mathrm{Ad}\, x\, \mathrm{Ad}\, y)$ or of the form determined by any representation of $\mathfrak{L}$. In general $\mathfrak{R}(\mathfrak{L}) = \{x \mid (x, \mathfrak{L}') = 0\}$, but it is shown that there is an extension Σ of Φ and an algebra $\mathfrak{M}$ over Σ such that (i) $\mathfrak{M} \supset \mathfrak{L}$ and $\mathfrak{R}(\mathfrak{M}) \supset \mathfrak{R}(\mathfrak{L})$ as Φ subalgebras, (ii) $\mathfrak{M}/\mathfrak{R}(\mathfrak{M}) \cong (\mathfrak{L}/\mathfrak{R}(\mathfrak{L}))_\Sigma$, (iii) $\mathfrak{R}(\mathfrak{M}) = \{x \mid (x, \mathfrak{M}) = 0\}$. A second, closely related, result is that $\mathfrak{L}$ has a faithful representation $x \to S_x$ in a space over an extension Σ of Φ such that $\mathfrak{R}(\mathfrak{L}) = \{x \mid \mathrm{tr}(S_x S_y) = 0$ for every y in $\mathfrak{L}\}$. *W. G. Lister* (Providence, R. I.).

Fleischer, Isidore. Sur les corps localement bornés. C. R. Acad. Sci. Paris 237, 546–548 (1953).

Let K be a topological division ring which is locally bounded and minimal in the sense that no weaker locally bounded topology exists. Suppose further that the multiplicative commutator subgroup of K is bounded. The author proves that the topology of K can be given by a valuation. Since division rings of type V are minimal and locally bounded, this improves the author's previous result that division rings of type V admit a valuation [same C. R. 236, 1320–1322 (1953); these Rev. 14, 720]. A number of further remarks are made, notably the following: any locally bounded topology can be weakened to a minimal one.

I. Kaplansky (Chicago, Ill.).

Theory of Groups

Croisot, Robert. Demi-groupes et axiomatique des groupes. C. R. Acad. Sci. Paris 237, 778–780 (1953).

In his thesis [Uppsala, 1953; these Rev. 15, 99], B. Stolt set himself the problem of determining, among all possible systems of axioms chosen from among a set of 74 axioms applicable to an arbitrary binary operation, which systems define a group. Eight systems were left undecided. Three of these contained both the axioms of existence and uniqueness of the product ab, as well as associativity (included in every system under consideration), and so constituted problems in the domain of demi-groups. The author solves these, finding that two of them define groups, while the third does not. The first of these yields the theorem that a demi-group D is a group if it contains an element d with the following two properties: (1) for every $a \in D$, the equation $xa = d$ is uniquely solvable for x; (2) there exists $c \in D$ such that the equation $cy = d$ has a unique solution y.

A. H. Clifford (Baltimore, Md.).

Tsuboi, Teruo. On the abelian factor group. Sci. Rep. Saitama Univ. Ser. A. 1, 1–8 (1952).

The author notes the well known result that a group in which the ascending central series climbs non-trivially above the center has non-trivial endomorphisms into its center. By using the transfer, he shows that if N is a normal subgroup of a group G where $[G:N]$ is relatively prime to the order of N, then G is mapped homomorphicly onto the abelian group $N/(N \cap G')$. (Here G' is the derivative of G.) Let G be a solvable group of order $g = mn$ where $(n, m) = 1$. By P. Hall [Zassenhaus, Lehrbuch der Gruppentheorie, Teubner, Leipzig-Berlin, 1937], there exists a normal subgroup N of G of order n. If N is included in the center of its

normalizer, it is proved that G contains a normal subgroup with N as a system of representatives. Hence N is an abelian endomorphic image of G of order n. *F. Haimo.*

Tsuboi, Teruo. Note on metabelian groups. Sci. Rep. Saitama Univ. Ser. A. 2, 59–62 (1953).

For a group of order p^n with normal abelian subgroup A of index p, Tuan [Acad. Sinica Science Record **3**, 17–23 (1950); these Rev. **12**, 800] proved that $A/(A\cap Z_i)\cong G_{i+1}$, $i=1, 2, \cdots, c$, where Z_i is the ith member of the ascending central series of G, G_{i+1} is the $(i+1)$-th member of the lower central series of G and c is the class of G. The author shows that the same conclusion holds for $i=1, 2, \cdots$, if G is any group (not necessarily finite), A is a normal abelian subgroup of G and G/A is a finite cyclic group. (The word "cyclic" has been omitted from the statement of Theorem 2 of the paper.) The method consists in showing that, for a fixed representative s of a generating coset of G/A, the mapping $f(x)=(x, s, s, \cdots, s)$, an $(i+1)$-fold commutator, is a homomorphism of A onto G_{i+1}. *F. Haimo.*

Tsuboi, Teruo. On finite simple groups. Sci. Rep. Saitama Univ. Ser. A. **2**, 55–57 (1953).

Let G be a simple group, and let H be a subgroup of G of order m. If $[G:H]$ is relatively prime to both m and $\varphi(m)$ (Euler's function), then the author proves that $\sqrt{m}$ is an upper bound for the orders of the elements in the center of H. He first proves that the cyclic subgroups of H have, under slightly less restrictive conditions, their normalizers coinciding with their centralizers. He then employs the transfers of G into the cyclic subgroups of the center of H. A corollary is the theorem of S. Tchounikhin [Math. Ann. **112**, 95–97 (1935)]: Let p be the minimum prime divisor of the order of a simple group G, where p^a divides that order but p^{a+1} does not. Let P be a Sylow subgroup belonging to the prime p. Then $\sqrt{p^a}$ is an upper bound for the orders of the elements of the center of P. *F. Haimo.*

Prokof'ev, A. N. On conditions for the number of solutions of the equation $X^n=1$ in a group to be minimum. Ukrain. Mat. Žurnal **4**, 427–430 (1952). (Russian)

According to a classical theorem of Frobenius the number of solutions of $X^n=1$ is a multiple of n. The author is concerned with the case where it is exactly n. Certain sufficient conditions had been given earlier by Fu [Quart. J. Math., Oxford Ser. **17**, 253–256 (1946); these Rev. **8**, 436]. The author gives a different type of sufficient condition, and, in the case where n is a power of a prime, gives necessary and sufficient conditions. These conditions refer in a rather complicated way to the behavior of the Sylow subgroups. *I. Kaplansky* (Chicago, Ill.).

Szele, Tibor. The multiplicative group of the roots of unity. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 55–58 (1953). (Hungarian)

A group G has property P if no two of its subgroups are isomorphic. The author proves that G has property P if and only if it is isomorphic with some subgroup of the multiplicative group of all roots of unity. *P. Erdős.*

Kertész, A., and Szele, T. Abelian groups every finitely generated subgroup of which is an endomorphic image. Acta Sci. Math. Szeged **15**, 70–76 (1953).

Let G be a torsion abelian group. The authors prove that every finitely generated subgroup of G is an endomorphic image of G if and only if each primary component P of G which has a non-trivial maximum complete subgroup A has the property that in P/A there is no bound to the order of the elements. On the other hand, suppose that G is an abelian group with some elements of infinite order. A necessary and sufficient condition that every finitely generated subgroup of G be an endomorphic image is that G contain a free abelian group of rank n as a direct summand for each positive integer n or that G splits into the direct sum of a free abelian group of finite rank and a torsion abelian group H, where every finitely generated subgroup of H is an endomorphic image of H. *F. Haimo* (St. Louis, Mo.).

Mišina, A. P. Some conditions for splitting of mixed Abelian groups. Ukrain. Mat. Žurnal **3**, 218–232 (1951). (Russian)

[For reference, see Baer, Ann. of Math. (2) **37**, 766–781 (1936).] Let A be a mixed abelian (additive) group with maximal periodic subgroup F; let $\mathfrak{F}^*=A/F$; let F_2 be the primary component of F for the prime 2. The problem is to investigate what conditions will insure the converse of the result: if A splits (that is, if $A=F+\mathfrak{F}$, where $\mathfrak{F}\cong\mathfrak{F}^*$), then every pair of automorphisms, one of F and the other of $\mathfrak{F}^*$, is induced by a suitable automorphism of A. Denote by ψ_0 the automorphism of F which maps each element into its negative. The group A splits if and only if A has an automorphism ϕ which induces ψ_0 in F, which induces the identity map in $\mathfrak{F}^*$, and which furthermore satisfies the condition that for each $a\in A$, there exists $f\in F$ such that $\phi(a)=a+2f$. The author studies various restrictions imposed on either F or $\mathfrak{F}^*$ which yield assertion (E): if the pair of automorphisms, ψ_0 of F and the identity map of $\mathfrak{F}^*$, is induced by an automorphism of A, then A splits. For example, if $\mathfrak{F}^*=2\mathfrak{F}^*$, then A satisfies assertion (E). If F_2 is zero, then A satisfies (E). Indeed, the general problem may be reduced to the case $F=F_2$. If $\mathfrak{F}^*$ has finite rank and if $F=F_2$, then A satisfies (E). If $D(\mathfrak{F}^*)$ exists [cf. Baer's Definition 3; 2] and if $F=F_2$, then A satisfies (E). Other sufficient restrictions are also presented. *R. A. Good.*

Čunihin, S. A. On the imbedding and number of subgroups in π-separable groups. Doklady Akad. Nauk SSSR (N.S.) **91**, 461–462 (1953). (Russian)

[For terminology and notation, see Čunihin, same Doklady (N.S.) **86**, 27–30 (1952); these Rev. **14**, 350.] Let $p_1, p_2, \cdots, p_k, \cdots, p_n$ be distinct primes belonging to Π, let $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}>1$ be a Π-Sylow divisor of g, let $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ be maximal Π-Sylow divisor of g; and for each $j=1, \cdots, n$, let ω_j be the least positive exponent such that $p_j^{\omega_j}$ divides indices of a composition series for $\mathfrak{G}$. The number of indices of a composition series of an arbitrary finite group which are not relatively prime to the natural number r is called the r-composition length of the group. Each of the three following conditions is individually sufficient in order that every subgroup of order m_1 of the Π-separable group $\mathfrak{G}$, where m_1 divides m, be contained in at least one subgroup of order m: (i) if the m-composition length of $\mathfrak{G}$ is k; (ii) if m_1 is relatively prime to m/m_1; (iii) more generally, if $m_1=p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$ with each $\beta_i>0$, and if for each $i=1, \cdots, k$, the group $\mathfrak{G}$ either is p_i-solvable or satisfies the relation $(\alpha_i-\beta_i)(\alpha_i-\omega_i)=0$. The number ρ of subgroups of order m in the Π-separable group $\mathfrak{G}$ divides g/m and moreover, for every prime $p_j\in\Pi$ such that either $\mathfrak{G}$ is p_j-solvable or $\omega_j=\alpha_j$, the highest power of p_j dividing ρ is a divisor of an index of the chief series of $\mathfrak{G}$ and is congruent to one modulo a suitable prime divisor of m. *R. A. Good* (College Park, Md.).

Čarin, V. S. On the minimality condition for normal divisors of locally soluble groups. Mat. Sbornik N.S. **33(75)**, 27–36 (1953). (Russian)

The question is: for what classes of groups does the minimal condition for normal subgroups imply the minimal condition for subgroups? Results by Jennings [Bull. Amer. Math. Soc. **50**, 759–763 (1944); these Rev. **6**, 114], Ado [C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 471–473 (1946); **58**, 523–524 (1947); these Rev. **8**, 437; **9**, 409], and Černikov [ibid. **58**, 1287–1289 (1947); these Rev. **9**, 492] are extended: the implication holds for locally nilpotent groups. Having shown previously [ibid. **66**, 575–576 (1949); these Rev. **10**, 677] that the implication fails for solvable groups, the author now shows that the implication holds for locally solvable groups such that each factor of some chief series of the group has finite rank. *R. A. Good.*

Sesekin, N. F. On locally nilpotent groups without torsion. Mat. Sbornik N.S. **32(74)**, 407–442 (1953). (Russian)

The author studies properties of a group G which is torsion-free, is locally nilpotent, and contains a maximal abelian subgroup A of finite rank r. Each of the four following conditions is individually sufficient in order that G be nilpotent and have finite rank: (i) that A be contained in a nilpotent normal subgroup; (ii) that G contain a maximal abelian characteristic (or invariant) subgroup of finite rank; (iii) that G be a group with category [Kontorovič, Mat. Sbornik N.S. **28(70)**, 79–88 (1951); these Rev. **12**, 671]; (iv) that G be an N-group. A group is called an N-group and is said to be covered by nilpotent normal subgroups provided each of its elements belongs to such a subgroup. Additional results are found for the special case $r=2$. If a torsion-free metabelian group M has a maximal abelian subgroup of finite rank $n+k$, where n is the rank of an isolated subgroup Z contained in the center of M and containing the commutator subgroup of M, then the rank of M/Z does not exceed $k(n+1)$. A torsion-free N-group having a maximal abelian normal subgroup of finite rank n has rank not exceeding $n(n+1)/2$. If a subgroup H of finite rank n of an R-group F [Kontorovič, ibid. **22(64)**, 79–100 (1948); these Rev. **9**, 493] has an ascending central series relative to F, then the quotient-group of F modulo the centralizer of H has rank not exceeding $n(n-1)/2$. Other results concern the so-called h-group, a non-abelian locally nilpotent group which is a locally cyclic extension of a torsion-free abelian group. The paper concludes with two examples, one of a torsion-free group of class $\omega 2+1$ with a maximal abelian subgroup of rank 3, the other of a torsion-free N-group without center. *R. A. Good* (College Park, Md.).

Knoche, Hans-Georg. Über den Frobeniusschen Klassenbegriff in nilpotenten Gruppen. II. Math. Z. **59**, 8–16 (1953).

A p-group is said to be of breadth β if p^β is the greatest number of elements which lie in one class of conjugate elements. Groups of breadth 0 are therefore Abelian. Groups of breadth 1 have commutator subgroup of order p and have been discussed by the author in a previous paper [Math. Z. **55**, 71–83 (1951); these Rev. **13**, 721]. It is shown that p-groups of breadth 2 are of class ≤ 3 while p-groups of breadth 3 are of class ≤ 4. Groups of breadth 2 are studied in some detail. It is shown that any p-group of breadth 2 has Abelian φ-group and commutator group of order $\leq p^3$. *D. C. Murdoch* (Vancouver, B. C.).

Szép, J., und Rédei, L. Eine Verallgemeinerung der Remakschen Zerlegung. Acta Sci. Math. Szeged **15**, 85–86 (1953).

Let G be a group with double chain condition for normal subgroups and with a (possibly empty) operator domain. The authors prove the following obvious generalisation of the Remak-Krull-Schmidt theorem. If G is decomposed in the form $G=N_1N_2\cdots N_n$, where the N_i are normal subgroups of G, no N_i is superfluous, and no N_i is the product of two proper normal subgroups, we put $N_i\cap N_k=D_{ik}$ and $\prod_{i\neq k}D_{ik}=D$. If another decomposition $D=M_1M_2\cdots M_m$ of the same type is given in which the group corresponding to D coincides with D, then $m=n$ and after a suitable numbering N_iD/D is centrally isomorphic to M_iD/D in G/D. Moreover $G=N_1N_2\cdots N_iM_{i+1}\cdots M_n$ for $i=1, 2, \cdots, n$. *K. A. Hirsch* (London).

Huppert, Bertram. Über die Auflösbarkeit faktorisierbarer Gruppen. Math. Z. **59**, 1–7 (1953).

If a finite group G is the product of two subgroups H and K, $G=HK$, then G turns out to be soluble under suitable restrictions on H and K. Solubility of the factors is not enough, as the simplest examples show. The affirmative statement in the case of two prime-power groups is Burnside's classical result. The best results so far are the theorems by Itô [Acta Sci. Math. Szeged **14**, 83–84 (1951); these Rev. **14**, 13] and Gruenberg [Proc. Cambridge Philos. Soc. **48**, 202 (1952); these Rev. **14**, 14]: The product of a nilpotent group with an abelian group or with a p-group is soluble. The author has recently [Math. Z. **58**, 243–264 (1953); these Rev. **14**, 1059] given a simplified proof for the case of two abelian groups. Refining his arguments, he now proves the following three theorems: The product of a dihedral group with an abelian group or with a p-group or with another dihedral group is soluble. The ingenious proofs proceed by induction on the order of the group and use, apart from the theorems mentioned above, a surprisingly large group-theoretical apparatus. *K. A. Hirsch.*

Bachmann, Friedrich. Eine Kennzeichnung der Gruppe der gebrochen-linearen Transformationen. Math. Ann. **126**, 79–92 (1953).

The main purpose of this paper is to give, in terms of involutory elements, necessary and sufficient conditions for an abstract group G to be isomorphic with the group of all homographic transformations: (*) $X'=(AX+B)(CX+D)^{-1}$ with $AD-BC\neq 0$, on an arbitrary field of characteristic $\neq 2$. Let Greek (resp. Latin) letters denote arbitrary (resp. involutory) elements of G, and call "joinable" (verbindbar) every two elements α, β of G such that there exists $a, b, c \in G$ with $\alpha=ac$ and $\beta=bc$. Then these conditions are stated as follows. Fundamental Axiom: Every element of G is the product of two involutory elements of G. Ax. 1: If $a\neq b$ and if abc and abd are involutory, acd is involutory. Ax. 2: There exist $a, b \in G$ which are not joinable. Ax. 3: If neither a, b, nor c, d are joinable, ab, cd are joinable. Ax. 4: If neither a, b, nor a, c, nor a, d are joinable, abc or abd or acd is involutory. On the basis of these axioms pencils of involutions and singular pencils of involutions, or points, are defined. The main part of the proof consists in introducing field operations on the set of all points in such a way that the elements of G, considered as acting on this set, are the homographic transformations (*). In case G is viewed as the group of all projectivities leaving fixed a given conic section in a projective plane, the axioms and the notions

of pencils and of points have very simple geometrical meaning.

If one replaces in the axioms above, Ax. 2, 3 and 4 by the negation of Ax. 2, and the Fund. Ax. by the following: "G is generated by its involutory elements, and no such element is permutable with all elements in G", one gets a set of axioms which characterizes the motion groups of elliptic geometries; this is proved by using A. Schmidt's axioms for these groups [J. Reine Angew. Math. **186**, 230–240 (1949); these Rev. **12**, 524]. *J. Tits* (Brussels).

Niggli, Alfred. Charakterentafeln als Ausdruck der Symmetrieeigenschaften von Molekülen und Kristallen. Schweiz. Mineral. Petrog. Mitt. **33**, 21–113 (1953).

Der erste Teil der Arbeit gibt eine ausführliche Darstellung der Punktsymmetrielehre mittels der von Pólya eingeführten Symmetrieformel. Teil II und III erläutern die komplexere Symmetrielehre der Raumgruppen mittels translativer und rotativer Charaktere. Dabei wird in der Kristallographie das Wort Charakter in einem von der Gruppentheorie abweichendem Sinn verstanden. In der Kristallographie wird unter dem Charakter einer Symmetrieoperation die zusätzliche Verschiebung (translativer Charakter) oder Drehung (rotativer Charakter) des Formelementes gegenüber seiner Lage in der starren Punktsymmetriegruppe verstanden. Teil IV behandelt die allgemeinen Symmetriesätze, während im V. Teil die Charaktere für physikalische Eigenschaften gegeben werden.

J. J. Burckhardt (Zürich).

Littlewood, D. E. On the Poincaré polynomials of the classical groups. J. London Math. Soc. **28**, 494–500 (1953).

The author improves upon Murnaghan's derivation of generating functions for the Poincaré polynomials of the classical groups [Proc. Nat. Acad. Sci. U. S. A. **38**, 608–611 (1952); these Rev. **14**, 134]. In the case of the full linear group the results depend on two preliminary theorems: if $\{\lambda\}$ and $\{\lambda\}'$ denote S-functions of $\alpha_1, \cdots, \alpha_m$ and $\beta_1, \cdots, \beta_n$ respectively, then (i) $1/\pi(1-\alpha_i\beta_j)=1+\sum\{\lambda\}\{\lambda\}'$; (ii) $(1+\alpha_i\beta_j)=1+\sum\{\lambda\}\{\tilde{\lambda}\}'$, where $(\tilde{\lambda})$ is conjugate to (λ). In the other cases the generating functions are obtained by considering known properties of S-functions under the multiplication $\otimes$. *G. de B. Robinson.*

Tits, J. Généralisations des groupes projectifs basées sur leurs propriétés de transitivité. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **27**, no. 2, 115 pp. (1952).

A group G of transformations of a set E is here called n-transitive, if for any two ordered n-tuples of points of E there is exactly one element of G sending the first into the second. The author has previously studied triply transitive groups and their connection with projective geometry [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 224–233, 568–589 (1949); these Rev. **11**, 9, 320]. The main topic of the present monograph is the "almost n-transitive" group: there exist ordered n-tuples of points of E (called "non-singular") which are left fixed only by the identity of G, and G is transitive over the set of non-singular n-tuples. Example: the projective group in projective $(n-2)$-space. Chapter I defines and studies basic concepts, suggested by analogy with projective groups. E.g.: The points $a_1, \cdots, a_k$ are independent if they are contained in a non-singular n-tuple. The singular subspace of rank k determined by them consists of all points which together with the a_i form a singular $(k+1)$-tuple, i.e., one not contained in a non-singular n-tuple. Groups are classified according to homogeneity properties and behavior of singular subspaces. E.g., a group is of hyperprojective type, if it is homogeneous in a certain sense, its subspaces of rank 1 are points and its subspaces of rank $n-1$ are trivial, i.e., finite union of subspaces of rank $n-2$, in a specified fashion.

Chapter II studies triply transitive groups, mainly by considering the involutions in the group. An algebraic formalism is set up, related to Zassenhaus' Fastkörper [Abh. Math. Sem. Hamburg. Univ. **11**, 187–220 (1935)], which allows to consider the transformations as fractional linear transformations in one variable. From this it is shown that a group is the projective group of a projective line if (it is triply transitive and) any two pairs of points determine an involution or any two hyperbolic transformations with the same fixed points commute, etc.

Chapter III treats groups of projective type and dimension >1. Their properties are studied in detail, and the projective groups characterized among them. Chapter IV studies n-transitive groups for $n>3$ (omitting the symmetric group on n objects). There is always the symmetric group on $n+1$ and the alternating group on $n+2$ objects. It is shown that besides these there exist only 4-transitive groups on a set E of 11 points, 5-transitive groups on a set E of 12 points, and no other n-transitive groups for $n>5$; it should be noted that E is not assumed to be finite.

Chapter V studies hyper-projective groups, for dimension >1. It is shown that, for $\dim>2$, all such groups are of projective type except for a finite number of groups, operating on finite sets. This is achieved by a detailed study of the singular subspaces.

Chapter VI studies groups which have "moving frames" in the sense that there exist n-tuples of points which are left fixed only by the identity. Several concepts are introduced which the author plans to use for a general study of transformation groups: a "sphere" is essentially the orbit of a point under all elements of the group which leave a given finite set, the "center", fixed. An invariant is a partition of E into sets, invariant under G; the invariants form a partially ordered set. The concepts are illustrated with several examples. *H. Samelson* (Princeton, N. J.).

Van Hove, Léon. Sur certaines représentations unitaires d'un groupe infini de transformations. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **26**, no. 6, 102 pp. (1951).

The group Γ in question is that of all C^∞ homeomorphisms g of $(2n+1)$-dimensional euclidean space E into itself which leaves invariant the Pfaffian form $\omega=ds-\sum p_i dq_i$, in the $2n+1$ real variables s, $p_1, \cdots, p_n, q_1, \cdots, q_n$. The representations in question are first, the canonical representation R on the space of all complex-valued Lebesgue square-integrable functions f on E, given by the equation $(R(g)f)(x)=f(g^{-1}(x))$, and second, the representations R_a, $-\infty<a<\infty$, that are obtained by decomposing R (as a direct integral of the R_a) relative to the operations of translations of s, the p's and q's being held fixed, or alternatively by taking the Fourier transform with respect to the variable s. The main results of the paper are: (1) the irreducibility of the representations R_a; (2) the essential hypermaximality of the infinitesimal generators of the representations R and R_a, when restricted to certain convenient domains; (3) applications to the correspondence principle in quantum mechanics [summarized in Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 610–620 (1951); these Rev. **13**, 519]. This is the

first treatment of the unitary representations of an infinite (-dimensional) Lie group.

In part these developments grow out of the well-known operational approach to classical mechanics due to Koopman and von Neumann, and in particular these authors made use of the representation R_0. The differential of this representation takes the p_i and q_j into commuting operators, while for $a \neq 0$, they are taken into operators P_i and Q_j for which $[P_i, Q_j] = ia^{-1}$. By virtue of this the representation R_a with $a = \hbar^{-1}$ affords an interesting scheme for going from a classical Hamiltonian to a quantum-mechanical one in which brackets are preserved but certain significant desirable properties are lacking. The proofs of (1) and (2) above require considerable careful analysis. *I. E. Segal.*

Gel'fand, I. M., and Graev, M. I. Unitary representations of the real unimodular group (principal nondegenerate series). Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 189–248 (1953). (Russian)

The authors present a series of continuous unitary representations on Hilbert space of the real unimodular group (=group of space matrices of determinant unity) of arbitrary order, and prove their irreducibility. In the case of the 2×2 group, all continuous unitary irreducible representations were obtained by Bargmann [Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 133] by infinitesimal methods. The present series of representations, called the principal non-degenerate series, is obtained by global methods parallel to those used by Gelfand and Neumark in determining all the continuous unitary irreducible representations of the complex unimodular group [Trudy Mat. Inst. Steklov. **36** (1950); these Rev. **13**, 722]. In particular, all the present representations are of multiplier type, and they do not exhaust the representations (of the stated type) of the real unimodular groups. *I. E. Segal.*

***Fantappiè, Luigi. Gli operatori funzionali vettoriali e tensoriali, covarianti rispetto a un gruppo qualunque.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 99–104. Casa Editrice Perrella, Roma, 1953.

A general discussion of the role of operators and Lie groups in a quantum-mechanical universe. *I. E. Segal.*

Borel, Armand. Les bouts des espaces homogènes de groupes de Lie. Ann. of Math. (2) **58**, 443–457 (1953).

The paper deals with n-dimensional varieties M on which a Lie group G operates transitively; M is of the form G/H, H is the isotropy group and is closed. In the first part of the paper it is shown that M has at most two ends, in the sense of Freudenthal's celebrated theorem, if H is connected. However, the conformal group of the interior of a unit circle is shown to have coset-spaces with more ends, or with infinitely many ends for an H not connected. The principal tool in part I is based on the fundamental theorem of E. Cartan-Malcev-Iwasawa and is as follows. Theorem 1. If H is a closed connected subgroup of a connected Lie group G, and if K and L, $K \supset L$, are maximal compact subgroups of G and H, G being homeomorphic to $K \prod R^s$ and H to $L \prod R^t$, then for every integer i the cohomology group $H^{i-s+t}(K/L)$ is isomorphic to the cohomology group with compact supports $H_c^i(K/L \prod R^{s-t})$ and this, in turn, to $H_c^i(G/H)$.

In part II, the author studies spaces $M = G/H$ on which G is "m-transitive in the larger sense": given two sets of m (or fewer) distinct ordered points of M, there is at least one element of G carrying one of these m-sets to the other. The space M in this situation is called m-homogeneous. The author shows that no simply-connected space can be compact and 4-homogeneous, or non-compact and 3-homogeneous. The simply connected 2-homogeneous spaces are the spheres, the complex and quaternion projective spaces, the projective octave-plane, and the euclidean spaces. Only the spheres are 3-homogeneous. A wide variety of techniques, not easily summarized, are used in part II.

L. Zippin (Flushing, N. Y.).

Matsushita, Shin-ichi. On topological groups. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **3**, 27–42 (1952).

Extensions of known results about free topological groups and the Tannaka duality theorem, by such techniques as postponing division by the kernels of the relevant homomorphisms, etc. *I. E. Segal* (New York, N. Y.).

Ellis, David, and Lang, Gaines. The space of groupoids over a compactum. Acta Math. **89**, 209–215 (1953).

For any two operations a, b (not necessarily continuous, it would seem) from $M \times M$ to M (M a compactum with metric d), there is defined a distance

$$D(a, b) = \sup\,(a(x, y), b(x, y)),$$

x, y ranging over M. Metric properties of the class G of all such $a, b, \cdots$ and of its subsets of semigroup operations, quasigroup operations, loop operations, etc., are established.

R. Arens (Princeton, N. J.).

NUMBER THEORY

Uhler, Horace S. On the 16th and 17th perfect numbers. Scripta Math. **19**, 128–131 (1953).

Garcia, Mariano. A generalization of multiply perfect numbers. Scripta Math. **19**, 209–210 (1953).

Palamà, G. Diophantine systems of the type

$$\sum_{i=1}^{p} a_i^k = \sum_{i=1}^{p} b_i^k, \quad (k = 1, 2, \cdots, n, n+2, n+4, \cdots, n+2r).$$

Scripta Math. **19**, 132–134 (1953).

Paasche, I. Ein zahlentheoretisch-logarithmischer "Rechenstab". Zum Satz von Moessner. Math. Naturwiss. Unterricht **6**, 26–28 (1953).

The theorem of Moessner was stated by him without proof [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 29 (1952); these Rev. **14**, 353] and first proved by Perron [ibid. **1951**, 31–34 (1952); these Rev. **14**, 353]. Other proofs have been given by H. Salié [ibid. **1952**, 7–11 (1953); these Rev. **14**, 846] as well as by the author of this note [ibid. **1952**, 1–5 (1953); these Rev. **14**, 846]. In this note the author comments on these proofs and gives various examples of the theorem when it is generalized so as to start with an

arbitrary sequence of numbers and is further generalized so as to involve a more general scheme of deleting numbers prior to summation. He remarks that sums are transformed into products by such a process (hence the title) just as Moessner's original case leads to powers and suggests that this fact may be useful in the construction of calculating machines. *H. W. Brinkmann* (Swarthmore, Pa.).

Duparc, H. J. A., and van Wijngaarden, A. A remark on Fermat's last theorem. Nieuw Arch. Wiskunde (3) **1**, 123–128 (1953).

In a recent paper [Portugaliae Math. **11**, 129–132 (1952); these Rev. **14**, 536] Obláth discussed lower bounds for x, y, and z in $x^n+y^n=z^n$ but failed to notice a relation of Barlow which allows one to replace these bounds by practically their nth powers. This fact is pointed out by the present authors. Since n exceeds 253747888 in the "first case" of Fermat's Last Theorem this advantage pays off lavishly. The authors show that for the first case $\log\log z>23.3$. A result of Inkeri gives similarly $\log\log x>23$.

D. H. Lehmer (Los Angeles, Calif.).

Chang, Fu-Hwa. On a series whose general term is given by

$$w_n=\binom{n}{1}+\binom{n-1}{2}+\binom{n-2}{3}+\cdots.$$

Acad. Sinica Science Record **5**, 45–49 (1952). (Chinese summary)

The general series

$$U_n^{(\lambda)}=\sum_{k=0}^{\infty}\binom{n-\lambda}{k+\lambda}$$

is a solution of the difference equation

$$U_{n+1}^{(\lambda)}=U_n^{(\lambda)}+U_{n-1}^{(\lambda)}+\binom{n}{k-1}. \tag{1}$$

For $k=0$ we have the Fibonacci series $U_n^{(0)}=u_n=1, 1, 2, 3, 5, \cdots$. The author considers the case $k=1$ which is the only other interesting case in which the final term of (1) is a constant. Denoting this series by $U_n^{(1)}=w_n=1, 2, 4, 7, 12, \cdots$, it is shown that $\Delta w_n=u_{n-1}$, so that the w's are the partial sums of the Fibonacci numbers. A few other properties are mentioned such as the fact that w_n is the arithmetic mean of w_{n+1} and w_{n-2}. There is an incomplete discussion of the parity of the w's which could be improved by noting that $w_{n+6}\equiv w_n \pmod 4$. *D. H. Lehmer.*

***Duparc, Herman Johan Arie. Divisibility properties of recurring sequences.** Thesis, University of Amsterdam, 1953. 96 pp.

This treatise deals with a general theory of linear recurring sequences with elements in a ring R in which there is unique factorisation and such that the residue set with any element of R as modulus is finite.

The work is in two parts. The first part lays the general groundwork of algebraic properties of R and an extension of R to include polynomial elements. The second part introduces recurring sequences W_n satisfying $W_{n+N}=\sum_{k=1}^{N}a_kW_{n+N-k}$ $(n=0, 1, \cdots)$ with initial values and coefficients in R. The emphasis is on the period and quasi-periods of W_n with respect to a modulus m. That is, the author investigates numbers c such that $W_{n+c}\equiv uW_n \pmod m$ where u belongs to a special set.

After a general discussion of the case in which R is the ring of rational integers, special cases are considered. These include the classical Lucas functions, and their extension by Lehmer, as well as third-order and higher recurrences based on the trinomial equation. There are several numerical examples. There is a bibliographical list of 15 titles. The names of M. Hall, T. A. Pierce, and P. Poulet are missing.

D. H. Lehmer (Los Angeles, Calif.).

Riesz, Marcel. Sur le lemme de Zolotareff et sur la loi de réciprocité des restes quadratiques. Math. Scand. **1**, 159–169 (1953).

The lemma referred to in the title is the following. Let p be an odd prime not dividing D. If the sequence $D, 2D, 3D, \cdots, (p-1)D$ is reduced modulo p so as to produce a permutation P_D of

$$(*) \qquad 1, 2, 3, \cdots, p-1,$$

then D is a quadratic residue of p or a non-residue according as P is even or odd. The author shows how P can be expressed as the product of two involutions by considering the congruence $u_1u_2\equiv D \pmod p$, that is, $P_D=T_D\cdot T_1$, where T_K is the permutation which carries the numbers u of (*) into the numbers K/u modulo p. This fact is used, together with Gauss' Lemma to give a variation of the third proof of Gauss of the Law of Reciprocity. The whole presentation is very elegant. *D. H. Lehmer.*

Mordell, L. J. Note on the linear symmetric congruence in n variables. Canadian J. Math. **5**, 433–438 (1953).

Consider the congruence

$$f=a_0+a_1\sum x_1+a_2\sum x_1x_2+\cdots+a_nx_1x_2\cdots x_n\equiv 0 \pmod p.$$

The author has conjectured that, excluding certain exceptional cases, the number of solutions of $f\equiv 0 \pmod p$ satisfies $N=p^{n-1}+O(p^{\frac{1}{2}(n-1)})$, where the constant implied in the O is independent of the a's. In the present paper he proves the conjecture for $n\leqq 4$. *L. Carlitz.*

Jakóbczyk, Franciszek. Les applications de la fonction $\lambda_g(n)$ à l'étude des fractions périodiques et de la congruence chinoise $2^n-2\equiv 0 \pmod n$. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **5** (1951), 97–138 (1953). (Polish and Russian summaries)

The function $\lambda_g(n)$ denotes the exponent of g modulo n, that is the least number h for which $g^h\equiv 1 \pmod n$, or what is the same, the length of the period when the rational $1/n$ is written to the base g. The author claims that the function $\lambda_g(n)$ has been neglected in comparison with other numerical functions because no notation has been proposed for it. Besides giving the fundamental properties of $\lambda_g(n)$ the author applies the function to two problems; one having to do with a certain skew-symmetric property mod $(g-1)$ of the digits in $1/n$, and the other concerning the "chinese congruence", $2^n\equiv 2 \pmod n$.

Comparing the results obtained with those already known, previous writers seem to have got along well enough without benefit of notation. The author conjectures that the Fermat and Mersenne numbers are square-free. *D. H. Lehmer.*

Carlitz, L. Kummer congruences and the Schur derivative. Amer. J. Math. **75**, 699–706 (1953).

Let p be a fixed prime and let $\{a_n\}$ be a sequence of rational numbers that are integral (mod p). Suppose that the Kummer congruence

$$(*) \qquad \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}a_p^{(p-s)k/(p-1)}a_{m+sk}\equiv 0 \pmod{p^{r\epsilon}}$$

is satisfied for $p^{e-1}(p-1)|b$ and $m \geqq re$, $e \geqq 1$, $r \geqq 1$. The author first proves that congruence (*) is implied by the weaker congruence obtained by replacing e by 1 and b by $p-1$. Now let a generalized Schur derivative be defined by means of

$$\Delta a_{p^m} = (a_{p^{m+1}} - a_p{}^{p^m} a_{p^m})/p^{m+1},$$

$$\Delta^r a_{p^m} = (\Delta^{r-1} a_{p^{m+1}} - a_p{}^{p^{m+r-1}} \Delta^{r-1} a_{p^m})/p^{m+1}.$$

Then the author shows that if the sequence $\{a_m\}$ satisfies (*) it follows that the numbers $\Delta^r a_{p^m}$, $1 \leq r \leq p-1$, are integral for all m. Moreover, the residues of these numbers (mod p^m) are specified. The method of proof is essentially that used in an earlier paper of the author [Pacific J. Math. 3, 321–332 (1953); these Rev. 14, 951]. In conclusion the author indicates some extensions of his results and gives applications to the coefficients of Jacobi elliptic functions and to Eulerian polynomials. *A. L. Whiteman.*

Carlitz, L. Some congruences of Vandiver. Amer. J. Math. **75**, 707–712 (1953).

Let $\{a_m\}$ be a sequence of rational numbers that are integral with respect to a fixed prime modulus p and assume that

$$(1) \quad \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} a_{m+s(p-1)} a_p{}^{r-s} \equiv 0 \pmod{p^r, p^m},$$

where $a_p{}'$ is integral (mod p). Let $\{b_m\}$ denote another sequence that satisfies (1) with $a_{m+s(p-1)} a_p{}^{r-s}$ replaced by $b_{m+s(p-1)} b_p{}^{r-s}$, where $b_p{}'$ is integral (mod p). The author first proves that if $\lambda + \mu \equiv 0 \pmod{p}$, then

$$(2) \quad a^m b^n (\lambda a^{p-1} b_p{}' + \mu b^{p-1} a_p{}')^r \equiv 0 \pmod{p^r, p^m, p^n},$$

where it is understood that the left member of (2) is to be expanded in full and $a^{m+(r-s)(p-1)}$, $b^{n+s(p-1)}$ replaced by $a_{m+(r-s)(p-1)}$, $b_{n+s(p-1)}$, respectively. The author then gives a generalization of this result which includes as a special case a congruence for the Bernoulli numbers due to Vandiver [Bull. Amer. Math. Soc. **43**, 418–423 (1937)]. Finally the author indicates a few applications of his results, particularly to the coefficients of Jacobi elliptic functions.

A. L. Whiteman (Princeton, N. J.).

Carlitz, L. Note on some partition formulae. Quart. J. Math., Oxford Ser. (2) **4**, 168–172 (1953).

Let $p_k(m)$ be the generalized partition function defined by $\prod_{n=1}^{\infty} (1-x^n)^{-k} = \sum_{m=0}^{\infty} p_k(m) x^m$. Employing the method of Bailey [same J. (2) **3**, 158–160 (1952); these Rev. **14**, 138] the author shows that identities such as

$$\sum_{m=0}^{\infty} p_3(3m+2) x^m = 9 \prod_{n=1}^{\infty} \frac{(1-x^{3n})^9}{(1-x^n)^{12}}$$

can be derived from the familiar formula

$$\wp'(u) = -\sigma(2u)/\sigma^4(u),$$

when the $\wp$-function is replaced by its Fourier series.

A. L. Whiteman (Princeton, N. J.).

Szekeres, G. Some asymptotic formulae in the theory of partitions. II. Quart. J. Math., Oxford Ser. (2) **4**, 96–111 (1953).

The author obtains an asymptotic formula for $P(n, k)$, the number of partitions of n into at most k parts, by applying the method of steepest descents to the integral in

$$P(n, k) = \frac{1}{2\pi i} \int F(w) w^{-n-1} dw,$$

where $F(w) = \prod_{r=1}^{k} (1-w^r)^{-1} = \sum_{n=0}^{\infty} P(n, k) w^n$. His formula is valid for all ranges of k and n (cf. second paragraph on p. 98) and therefore improves the result obtained in his previous paper [same J. (2) **2**, 85–108 (1951); these Rev. **13**, 210]; the present method is also considerably simpler.

For bounded n/k^2 the formula reads

$$(1) \quad P(n, k) = \frac{1}{2\pi} B_0^{-1/2} \beta^{-2} \exp\left\{ 2\beta \int_0^v \frac{t\,dt}{e^t - 1} - (v\beta + \tfrac{1}{2}) \times \log(1 - e^{-v}) + \frac{1}{2}\left(\frac{v}{e^v - 1} - 1 \right) \right\} \times [1 + B_1(v)\beta^{-1} + \cdots + B_{m-1}(v)\beta^{-m+1} + O(\beta^{-m})],$$

for any $m > 0$, where $B_0 = \int_0^v t^2 e^t (e^t - 1)^{-2} dt$, $|B_\mu(v)| \leqq C_\mu$, and v, β are determined from $v\beta = k$ and a complicated transcendental relation. In particular, β is of the exact order of $n^{1/2}$. When n/k^2 is not bounded, the formula is slightly different. For $k \geqq \delta n$, δ fixed and > 0, the first term of (1) coincides with the first term of the Hardy-Ramanujan formula for $p(n)$, the unrestricted partition function.

The author also investigates $p(n, k)$, the number of partitions of n into exactly k positive parts. Erdös [Bull. Amer. Math. Soc. **52**, 185–188 (1946); these Rev. **7**, 273] proved that the maximum of $p(n, k)$ occurs at $k_1 = cn^{1/2}L + o(n^{1/2})$, $L = \log(cn^{1/2})$. In the present paper this is improved to

$$k = cn^{1/2}L + c^2(\tfrac{3}{2} + \tfrac{3}{4}L - \tfrac{1}{4}L^2) - \tfrac{1}{2} + O(n^{-1/2} \log^4 n).$$

Similar results are obtained for $q(n, k)$, the number of partitions of n into exactly k unequal parts.

Presumably, the author's method should be available for a variety of partition problems. *J. Lehner.*

Ankeny, N. C., and Rogers, C. A. Acknowledgement for "A conjecture of Chowla". Ann. of Math. (2) **58**, 591 (1953).

See same Ann. (2) **53**, 541–550 (1951); these Rev. **12**, 804.

Gupta, Hansraj. A generalization of the Möbius function. Scripta Math. **19**, 121–126 (1953).

The function considered by the author is denoted by $v_r(n)$, and could be defined by the identity

$$\sum_{n=1}^{\infty} v_r(n) n^{-s} = \zeta(rs)/\zeta(s).$$

This function includes the function μ of Möbius for $r = 0$ the function λ of Liouville for $r = 2$ and the "unit function" $[1/n]$ for $r = 1$. The author generalizes some of the lemmas in the Selberg-Erdös-van der Corput proof of the Prime Number Theorem. For example,

$$\Big| \sum_{n \leq x} n^{-1} v_r(n) \Big| \leqq 1 + x^{-1/r}$$

$$- \sum_{n \leq x} g(n) = \sum_{k=1}^{\infty} \theta(x/K^n) + o(x),$$

where $g(n) = \sum_{d|n} v_r(d) \log d$. *D. H. Lehmer.*

Delange, Hubert. Sur le nombre des diviseurs premiers de n. C. R. Acad. Sci. Paris **237**, 542–544 (1953).

The author proves that the values of

$$(\omega(n) - \log\log n)/(\log\log n)^{\frac{1}{2}},$$

where $\omega(n)$ is the number of different prime divisors of n, are normally distributed. His proof depends on estimates for the moments $S_r(x) = \sum_{n \leq x} (\omega(n) - \log\log n)^r$. To this end the author considers $\prod_r(x) = \sum_{k \leq x, k \in S} k^{-1}$, where S is the

set of all products of r different primes. The latter sum can be estimated by using a tauberian theorem proved in a previous paper [same C. R. **232**, 589–591 (1951); these Rev. **12**, 497].

The author seems not to be acquainted with a paper of Erdös and Kac [Amer. J. Math. **62**, 738–742 (1940); these Rev. **2**, 42], where the same result was obtained from a general theorem on the distribution of the values of additive number-theoretic functions. He also overlooked Kac's report on probability methods in analysis and number theory [Bull. Amer. Math. Soc. **55**, 641–665 (1949); these Rev. **11**, 161], where several aspects of the present problem were discussed, and the idea to use the moments was suggested.

N. G. de Bruijn (Amsterdam).

Subba Rao, K. Representation of numbers as sums of k-th powers. Math. Student **21**, 49 (1953).

Put $m=2^{[(k+1)]}$, and let t be any rational number. The author proves that there exists an infinity of rational numbers $x_1, x_2, \cdots, x_m$ satisfying $t=\sum_{i=1}^{m} a_i x_i^k$, $a_i=\pm 1$.

P. Erdös (South Bend, Ind.).

Rodosskiĭ, K. A. On the number of L-functions having zeros in some rectangle. Ukrain. Mat. Žurnal **3**, 399–403 (1951). (Russian)

The author proves that for $\Delta<1$ the number of L-functions (formed by characters mod D) which have zeros in the rectangle $\Delta\leqq\sigma\leqq 1$, $|t-T|\leqq\frac{1}{2}$ is $O(\log^8 (DT)(D^2T)^{3(1-\Delta)/\Delta})$. Similar results were proved by Linnik [Mat. Sbornik N.S. **15(57)**, 3–12, 139–178 (1944); these Rev. **6**, 260]. The author's proof is surprisingly short.

H. Heilbronn.

Rodosskiĭ, K. A. On the least prime number in an arithmetic progression and the zeros of L-functions. Doklady Akad. Nauk SSSR (N.S.) **88**, 753–756 (1953). (Russian)

Using the results of the paper reviewed above, the author simplifies the proof of Linnik's theorem [see references cited above and Mat. Sbornik N.S. **15(57)**, 347–368 (1944); these Rev. **6**, 260] that the smallest prime $p\equiv l \pmod D$ satisfies for $(D, l)=1$ the inequality $p=O(D^c)$, where c is an absolute constant.

H. Heilbronn (Bristol).

Tatuzawa, Tikao. On the number of the primes in an arithmetic progression. Jap. J. Math. **21** (1951), 93–111 (1952).

The paper is concerned with the order of the remainder term in the asymptotic formula for $\pi(x, k, l)$ which denotes the number of primes $p\leqq x$ such that $p\equiv l \pmod k$, where $(k, l)=1$. The main result is that for $x\geqq\exp(c_1 \log k \log\log k)$

$$\pi(x,k,l)=\frac{1}{\varphi(k)}\int_2^x \frac{du}{\log u}+O\left(\frac{x^\beta}{\varphi(k)\log x}\right)+O\left(\frac{1}{\varphi(k)}x^{1-c_1/\Delta}\right)$$

where $\Delta=\text{Max}\{\log k, (\log x \log\log x)^{3/7}\}$, and β, which depends on k, is the possible simple real zero, lying in the interval $1-c_1/\log k<\sigma<1$, of some one of the L functions modulo k. Also, the O estimates are uniform in k. This result is a slight improvement of known results, and yields for $\pi(x)=$ the number of primes $\leqq x$,

$$\pi(x)=\int_2^x \frac{du}{\log u}+O\left[x\exp\left(-c_6\frac{(\log x)^{3/7}}{(\log\log x)^{3/7}}\right)\right].$$

H. N. Shapiro (New York, N. Y.).

Ožigova, E. P. Modification of the method of the "sieve of Eratosthenes" given by A. Selberg. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 119–124 (1953). (Russian)

The sieve method of A. Selberg [Norske Vid. Selsk. Forh., Trondhjem **19**, no. 18, 64–67 (1947); these Rev. **9**, 271] is presented with slightly more detail than in the original paper of Selberg. Several applications of the method are stated.

H. N. Shapiro (New York, N. Y.).

Ricci, Giovanni. Sul coefficiente di Viggo Brun. Ann. Scuola Norm. Super. Pisa (3) **7**, 133–151 (1953).

Let H denote the constant $\prod(1-(p-1)^{-2})$, where the product extends over all the odd primes. Hardy and Littlewood [Proc. Cambridge Philos. Soc. **19**, 245–254 (1919)] conjectured that if $N_k(n)$ is the number of prime pairs less than n, whose difference is k, then

$$N_k(n)\sim\frac{2Hn}{\log^2 n}\prod_{p\mid k}\left(\frac{p-1}{p-2}\right),$$

where p runs over the odd prime divisors of k. The present author considers some functions related to $N_k(n)$ (which he refers to as the "coefficients of Viggo Brun" [cf. Brun, C. R. Acad. Sci. Paris **168**, 544–546 (1919)] and derives by elementary methods a number of inequalities involving the constant H.

A. L. Whiteman (Princeton, N. J.).

Shapiro, Harold N. Iterates of arithmetic functions and a property of the sequence of primes. Pacific J. Math. **3**, 647–655 (1953).

The author investigates iterates of arithmetic functions of the forms

$$f(n)=\prod f(p_i)\{A(p_i)\}^{a_i-1},$$

where $n=\prod p_i^{a_i}$, $f(p_i)$ is an integer, $1<f(p_i)<p_i$, $A(p_i)$ is an integer $\leqq p_i$ for odd primes p_i, while $f(2)=1$, $A(2)=2$. Let K denote the class of all such functions. $C_f(n)$ is the least exponent k such that the kth iterate $f^k(n)=2$. It is proved that if $g(n)$ is a function of K such that $C_g(n)$ is maximal for all primes p, that is, $C_g(p)\geqq C_f(p)$ for all $f\in K$, then $g(3)=2$, $g(5)=4$ or 3, and, for $i>3$, $g(p_i)=p_{i-1}$. Here p_i denotes the ith prime. The proof is by induction.

R. A. Rankin (Birmingham).

Fjellstedt, Lars. On a class of Diophantine equations of the second degree in imaginary quadratic fields. Ark. Mat. **2**, 435–461 (1953).

The author shows that the classical treatment of Pell's equation can be extended to the equation $x^2-\delta y^2=\pm 1$, where δ is a fixed integer in an imaginary quadratic field and x, y are to be found. He proves the existence of a fundamental solution and also gives sufficient (but not necessary) criteria for a solution to the fundamental. Finally he applies the results to the equation $x^3-\delta y^3=\gamma$. The paper has similarities with A. Z. Val'fiš, Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 133–151 (1951) [these Rev. **14**, 537].

J. W. S. Cassels.

Vandiver, H. S. A development of associative algebra and an algebraic theory of numbers. II. Math. Mag. **27**, 1–18 (1953).

[For part I see Math. Mag. **25**, 233–250 (1952); these Rev. **14**, 348.] The main objective of this part is to adjoin zero, negative integers, and rational fractions to the set of natural numbers that were introduced in part I. Each of these adjunctions is carried out by considering congruences modulo a certain polynomial (or modulo a set of poly-

nomials) and then defining equality as corresponding to such a congruence. However, in doing this it is not possible here to use the usual notion of a transcendental adjunction and consequently the idea of an indeterminate (in terms of which the polynomials are defined) has to be introduced in a special way. The paper contains an addendum to part I in which the author comments on various criticisms, mostly dealing with what he calls a "combination", that have been submitted to him. *H. W. Brinkmann* (Swarthmore, Pa.).

Roquette, Peter. Arithmetischer Beweis der Riemannschen Vermutung in Kongruenzfunktionenkörpern beliebigen Geschlechts. J. Reine Angew. Math. **191**, 199–252 (1953).

This paper comprises the complete detailed account of a new proof of the Riemann hypothesis for function fields of one variable. A brief comparison of the author's method of proof with the original proof of A. Weil was already made in the review of the author's brief description of his approach [Arch. Math. **4**, 6–16 (1953); these Rev. **14**, 848]. As indicated [loc. cit.] the outstanding feature may be found in the combination of Deuring's divisor-theoretic version of the classical correspondence theory with a novel theory of discriminants of systems of isomorphisms. In this manner all work can be carried out on the product of the underlying function field by itself, or in fully equivalent terminology, of the given non-singular curve by itself. This means in turn that the author succeeds in getting along with the concepts and theorems belonging to the theory of rank one valuations (the Riemann-Roch theorem included). The author's presentation, based on the aforementioned background, is self-contained and leads quite rapidly to the proof of the positive definiteness of Weil's trace function. [Still another shortcut to Weil's theorem, using divisors one product varieties, and not the full machinery of the general intersection theory, is presented in the unpublished Chicago thesis of F. Quigley, "An elementary proof of the Riemann hypothesis in function fields."] *O. F. G. Schilling.*

Watson, G. L. Minkowski's conjectures on the critical lattices of the region $|x|^p+|y|^p\leq 1$. II. J. London Math. Soc. **28**, 402–410 (1953).

In part I of this paper [ibid. **28**, 305–309 (1953); these Rev. **14**, 1065] the author considered two special symmetrical lattices, and determined which of their determinants is the smaller. The critical determinant is $\min_p \Delta(P)$, where P runs through the lattices that have six points on the boundary and none inside (apart from the origin). In the present part the author shows that min (Δ_0, Δ_1) gives a local minimum of $\Delta(P)$ for all p. In a small region around $p_0=2.57\cdots$ both give local minima, so that there is an unsymmetrical maximum. This disproves a conjecture of Minkowski stating that the only local extremals are Δ_0 and Δ_1. A similar result was given by H. Cohn [Ann. of Math. (2) **51**, 734–738 (1950); these Rev. **11**, 716], but the present author indicates an essential error in Cohn's paper.

N. G. de Bruijn (Amsterdam).

Ap Simon, H. G. The critical lattices of the off-centre hypercube. Quart. J. Math., Oxford Ser. (2) **4**, 204–209 (1953).

There is exactly one critical lattice of the hypercube A: $|y_r-1|\leq c_r$, where $c_r>1$ $(r=1, \cdots, n)$ and it is given by $y_r=(c_r-1)\xi_r+2\sum\xi_i$, the ξ_i being integral. It is determined as the unique critical lattice of a convex hyperbody D which lies in A and the symmetric image of A with respect to the origin. The uniqueness is proved by first showing that the lattice provides a space-filling lattice-packing of $\frac{1}{2}D$.

L. Tornheim (Ann Arbor, Mich.).

***Chinčin, A. J. Řetězové zlomky. [Continued fractions.]** Přírodovědecké Vydavatelství, Praha, 1952. 104 pp. 32 Kčs.

Translation by K. Rychlík of "Cepnye drobi" [2d ed., Gostehizdat, Moscow-Leningrad, 1949; these Rev. **13**, 444].

ANALYSIS

Fuchs, László. On algebraic systems in which an operation taking mean values is defined. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 27–35 (1953). (Hungarian)

This paper is a shortened Hungarian version of a previous paper of the author [Acta Math. Acad. Sci. Hungar. **1**, 303–320 (1950); these Rev. **13**, 922]. *P. Erdős.*

Hosszú, Miklós. Contribution à la théorie de l'équation fonctionnelle de la bisymmétrie. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 335–342 (1953). (Hungarian. Russian and French summaries)

The author proves the following theorems: Assume that

(1) $M[m(X, u), n(y, v)]=N[m(x, v), n(y, u)]$

where all the functions are strictly monotonic and differentiable. Then there exist $f(x)$, $g(x)$, $h(y)$, $X(x)$, $Y(y)$, $H(x, y)$ all differentiable and strictly monotonic so that

(2) $M(x, y)=N(x, y)=H[X(x)+Y(y)]$,

(3) $m(x, y)=X^{-1}[f(x)+h(y)]$, $n(x, y)=Y^{-1}[g(x)+h(y)]$.

If

(4) $M[m(x, u), m(y, v)]=N[m(x, y), m(u, v)]$,

then

(5) $M(x, y)=N(x, y)=G[ah(x)+bh(y)+c]$,

(6) $m(x, y)=g[af(x)+bf(y)+c]$, $g(t)=h^{-1}(t)$,

where again the functions are strictly monotonic and differentiable. Conversely, functions of the form (2) and (3) satisfy (1) and functions of the form (5) and (6) satisfy (4). The problem of characterising the functions which satisfy (2) and (5) was raised by Aczél. These results can be considered as generalizations of Aczél's condition of bisymmetry [Bull. Amer. Math. Soc. **54**, 392–400 (1948); these Rev. **9**, 501]. *P. Erdős* (South Bend, Ind.).

Calculus

***Dubnov, Ya. S. Osnovy vektornogo isčisleniya. Čast' II. Lineĭnye funkcii vektora. Vektornyĭ analiz (teoriya poleĭ). Načala tenzornogo isčisleniya. [Foundations of vector calculus. Part II. Linear vector functions. Vector analysis (theory of fields). Elements of tensor calculus.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 415 pp. 10.35 rubles.

The first part of this textbook, which is the result of lectures repeatedly given at Moscow University, appeared in 1930, its fourth edition in 1950 [these Rev. **12**, 165]. The author has endeavored to make the second part independent of the first, so that it represents a textbook of its own. The

first chapter deals with linear vector functions, the second with point functions and their derivatives, the third with the principles of tensor calculus, and the fourth with integration theorems. The earlier chapters are written in the dyadic notation, the last chapter in tensor and vector notation; the exposition of Stokes' theorem is rather more careful than in many similar books. An appendix (pp. 308–363) presents an outline of the dual calculus in two- and three-dimensional central affine geometry; the duality consists in the opposition of vectors and "doublets" (ordered pair of lines, resp. planes). There are no less than 492 problems; the answers are at the end of the book. The figures are good.

D. J. Struik (Cambridge, Mass.).

*Manarini, Mario. **Diadi vettoriali ed applicazioni.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 526–529. Casa Editrice Perrella, Roma, 1953.

Viene introdotta un'omografia vettoriale, in generale degenere di prima specie, che viene chiamata "diade vèttoriale". Vengono stabilite le sue principali proprietà e viene illustrata la sua utilità per la determinazione di grandezze meccaniche di uso comune. *Author's summary.*

*Hildebrand, F. B. **Methods of applied mathematics.** Prentice-Hall, Inc., New York, N. Y. 1952. xi+523 pp. $7.75.

Chapter 1. Matrices, determinants, and linear equations. Chapter 2. Calculus of variations and applications. Chapter 3. Difference equations. Chapter 4. Integral equations.

Table of contents.

Rufener, E. Renten und Todesfallversicherungen höherer Ordnung. Mitt. Verein. Schweiz. Versich.-Math. **53**, 166–188 (1953).

A systematic study of the operators $J=\int_x^\infty d\xi$ and $S=\sum_x^\infty$ and their application to life insurance mathematics.

H. L. Seal (New York, N. Y.).

Theory of Sets, Theory of Functions of Real Variables

Hoheisel, Guido, und Schmidt, Jürgen. Über die Konstruktion einer gewissen totalen Ordnung in Bäumen. Arch. Math. **4**, 261–266 (1953).

Let E be a partially ordered set with the property that for each x in E the set $A(x)=\{y \mid y \leqq x, y \in E\}$ is simply ordered. Let M be a subset such that, for each x in M, $A(x)$ is a subset of M; M may be empty. Denoting the set of proper upper bounds to M by $U(M)$, for each a and b in $U(M)$ write "$a=b$" if there exists x in $U(M)$ so that $x \leqq a$ and $x \leqq b$. Whenever $U(M)$ is decomposed into at least two equivalence classes by "$=$", call M a node and the equivalence classes determined by M, branches. In the node $A(a) \cap A(b)$, where a and b are incomparable elements of E, let Z_{ab} be the branch containing a. Theorem: Suppose that the set of all possible branches of E is partially ordered in some manner so that for each node M, the set of all branches determined by M is simply ordered. For a and b in E define $a \leqq' b$ when either $a \leqq b$, or when a and b are incomparable but $Z_{ab} \leqq Z_{ba}$. Then E is simply ordered under $\leqq'$ [a slightly stronger result is actually proved, namely that the conclusion is true if for each node M, the set of all branches determined by M is simply ordered in some manner].

S. Ginsburg (Coral Gables, Fla.).

Łoś, J. Recherches algébriques sur les opérations analytiques et quasi-analytiques. Ann. Soc. Polon. Math. **25** (1952), 131–139 (1953).

Given a family $\mathbf{X}$ of subsets of a set E and an ordinal α, an operation Φ of type α on $\mathbf{X}$ is said to be analytic if, for any $p, q \in E$ and $X_1, X_2, \cdots, X_\xi, \cdots, Y_1, Y_2, \cdots, Y_\xi, \cdots \in \mathbf{X}$, the conditions

$$p \in \Phi(X_1, X_2, \cdots, X_\xi, \cdots) \text{ and } q \text{ non-}\in \Phi(Y_1, Y_2, \cdots, Y_\xi, \cdots)$$

jointly imply that, for some $\xi<\alpha$, either $p \in X_\xi$ and q non-$\in Y_\xi$ or else p non-$\in X_\xi$ and $q \in Y_\xi$. For Φ to be quasi-analytic this is required to hold only in the special case in which $p=q$. An algebra $\Gamma=\langle A, \varphi\rangle$ is said to be quasi-analytic (analytic) if it is isomorphic to a system $\langle \mathbf{X}, \Phi\rangle$ where Φ is a quasi-analytic (analytic) operation over the family $\mathbf{X}$ of sets. It is shown that Γ is quasi-analytic (analytic) if and only if it is isomorphic to a subalgebra of a direct product of (pairwise isomorphic) two-element algebras.

B. Jónsson (Providence, R. I.)

Borsuk, K. An application of the theorem on antipodes to the measure theory. Bull. Acad. Polon. Sci. Cl. III. **1**, 87–90 (1953).

If f maps the n-ball, Q^n, into the metric space M considered partitioned into $k+1$ sets, $\{M_i\}$, and if $\{A(p) \mid p \in f(Q)\}$ is a family of measurable sets, suitably restricted, a typical result is: If $k<n$ then for some p_0, the measures of $\{A(p_0) \cap M_i\}$ are proportional to the measures of $\{A(p_0) \cap M_i\}$. *D. G. Bourgin* (Urbana, Ill.).

Takeda, Ziro. Perfection of measure spaces and W^*-algebras. Kōdai Math. Sem. Rep. **1953**, 23–26 (1953).

The author gives an alternative proof of the result that there exists a perfect measure space equivalent to any given localizable space [Segal, Amer. J. Math. **73**, 275–313 (1951), Th. 6.1; these Rev. **12**, 809] and makes some applications. *I. E. Segal* (New York, N. Y.).

Segal, I. E. Correction to "A non-commutative extension of abstract integration". Ann. of Math. (2) **58**, 595–596 (1953).

See same Ann. (2) **57**, 401–457 (1953); these Rev. **14**, 991.

Burkill, J. C., and Gehring, F. W. A scale of integrals from Lebesgue's to Denjoy's. Quart. J. Math., Oxford Ser. (2) **4**, 210–220 (1953).

Let D_1 and D_0 be the classes of real functions $f(x)$ $(a \leqq x \leqq b)$, integrable in the Lebesgue and special Denjoy senses, respectively. The authors study the class D_α $(0<\alpha<1)$ of functions which belong to D_0 and have integrals of bounded variation or order $1/\alpha$ in the sense of Wiener and the reviewer [Young, Acta Math. **67**, 251–282 (1936); see also Love, J. London Math. Soc. **26**, 1–13 (1951); these Rev. **12**, 599]. Applications are given in the classical theory of Fourier series as regards Cesàro summability, order of magnitude of coefficients, term-by-term integration.

L. C. Young (Madison, Wis.).

Goffman, Casper. One-one measurable transformations. Acta Math. **89**, 261–278 (1953).

Let $f(x)$ be a function defined on a unit cube in n-dimensional Euclidean space and taking values in a unit cube in m-dimensional Euclidean space. It is measurable if both f and f^{-1} transform Borel subsets into measurable subsets. After establishing a number of results regarding the extension of homeomorphisms defined over a certain type of zero-

dimensional set, the author proves a series of theorems which connect measurable transformations with homeomorphisms. An example is the following. If $f(x)$ is a one-one measurable transformation between open n-dimensional and m-dimensional unit cubes I_n, I_m, where n, m are any positive integers $\geqq 2$, then, for every $\epsilon>0$, there is a closed set $S\subset I_n$ of n-dimensional measure greater than $1-\epsilon$ and a closed set $T\subset I_m$ of m-dimensional measure greater than $1-\epsilon$ such that $f(x)$ is a homeomorphism between S and T. The author also shows that if $n=m=1$ then there exists a one-one transformation $g(x)$ such that $g(x)$ is of Baire class at most 2 and $g(x)=f(x)$ almost everywhere. It is not known whether an analogous result holds for higher-dimensional spaces. [Reviewer's remark: Lemma 4 is not correct, but it seems that it can be rectified.]

H. G. Eggleston (Cambridge, England).

*Cafiero, Federico. **Sulle famiglie compatte di funzioni additive di insieme astratto.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 30–40. Casa Editrice Perrella, Roma, 1953.

A family of additive set functions $\{\varphi(I)\}$ defined on an additive class of sets F is said to be equi-continuous if, given $\epsilon>0$ and $I_n\to 0$, there is an m such that $|\varphi(I_n)|<\epsilon$ for $n>m$ and every φ. The author gives two different proofs of the following generalization of a theorem of Vitali-Lebesgue: A sequence $\varphi_n(I)$ which converges on every $I\in F$ is equi-continuous. The following more general theorem is stated without proof: $\{\varphi_n(I)\}$ is equi-continuous if and only if, given a sequence of disjoint sets $\{I_r\}$ and an $\epsilon>0$, there is a ρ and an m such that $|\varphi_n(I_\rho)|<\epsilon$ for $n>m$.

M. Cotlar (Buenos Aires).

Pauc, Christian. Contributions à une théorie de la différentiation de fonctions d'intervalle sans hypothèse de Vitali. C. R. Acad. Sci. Paris **236**, 1937–1939 (1953).

Let μ be a finite measure defined on the Boolean σ-algebra $\mathfrak{B}$ of subsets of a set R ($R\in\mathfrak{B}$). Let $\mathfrak{R}$ be a system of finite partitions $\mathfrak{T}$: $R=J_1+J_2+\cdots+J_n+N$ where the $J_k\in\mathfrak{B}$ with $\mu(J_k)>0$ $(k=1, 2, \cdots, n)$ and $\mu(N)=0$. The norm $\gamma(\mathfrak{T})=\max\mu(J_k)$. The sets J_k are called "intervals" and their family is designated by $\mathfrak{J}$. Then $\mathfrak{B}^0$ denotes the minimal Boolean σ-algebra with unity R which contains $\mathfrak{J}$, and μ^0 means the complete extension of $\mu|\mathfrak{B}^0$ (the restriction of μ to $\mathfrak{B}^0$). It is assumed that $\mathfrak{R}$ be a directed system with respect to the fineness of the partitions $\mathfrak{T}$ and that there exist $\mathfrak{R}$-partitions of arbitrarily small norm. Four theorems are stated by the author (without proofs). I. If the interval function ψ is a.c. (absolutely continuous), then the upper and lower Burkill integrals of ψ can be represented as μ^0-integrals. II. Every (finitely) additive function ψ of bounded variation can be represented in one and only one manner as the sum of an a.c. interval function ψ_r and an additive function ψ_s which is singular in the sense of S. Saks [Theory of the integral, 2nd ed., Warszawa, 1937, p. 93]. The author gives an example of a μ-singular set function whose restriction to $\mathfrak{J}$ is not singular any more.

The "$\mathfrak{T}$-derivative" of an interval function ψ is the function defined on $R-N$ by $D(\psi, \mathfrak{T})(x)=\psi(J_k)/\mu(J_k)$ where J_k designates the $\mathfrak{T}$-interval containing the point x. If the functions $D(\psi, \mathfrak{T})$, directed with respect to the fineness of $\mathfrak{T}$, converge "in the mean" [i.e. in the space $L(\mu, R)$] to a limit $D(\psi)$, then $D(\psi)$ is called the "L-derivative" of ψ and ψ is said to be "L-derivable" on R with respect to $\mathfrak{R}$. III. In order that the a.c. interval function ψ be summable on R in the sense of Burkill, it is necessary and sufficient that ψ be L-derivable on R. IV. Under the assumption of an additional "axiom U" [see Haupt and Pauc, Arch. Math. **1**, 23–28 (1948), p. 25; these Rev. **10**, 359], for every a.c. function ψ the notions of the Burkill integral and the L-derivative defined according to the fineness either of the partition or of the norm are equivalent. *A. Rosenthal.*

Dezin, A. A. On imbedding theorems and the problem of continuation of functions. Doklady Akad. Nauk SSSR (N.S.) **88**, 741–743 (1953). (Russian)

Problems of imbedding and continuation of functions, similar to those considered by Nikol'skiĭ [Trudy Mat. Inst. Steklov. **38**, 244–278 (1951); these Rev. **14**, 32] are attacked by more elementary methods, but with somewhat weaker results. The technique is based on use of the averaging

$$f_h(P)=(2h)^{-n}\int_{x_1-h}^{x_1+h}\cdots\int_{x_n-h}^{x_n+h} f(Q)dv_Q$$

and the theory of L_p spaces. *M. G. Arsove.*

Amanov, T. I. On the imbedding theorem for differentiable functions of several variables. Doklady Akad. Nauk SSSR (N.S.) **88**, 5–8 (1953). (Russian)

Nikol'skiĭ [Trudy Mat. Inst. Steklov. **38**, 244–278 (1951); these Rev. **14**, 32] introduced the class

$$H_p^{(r)}(M_1, \cdots, M_n),\ (r)\equiv(r_1, \cdots, r_n),$$

of functions of n variables and proved the following theorem. Let $1\leqq p\leqq p'\leqq\infty$, $1\leqq m\leqq n$, $r_i>0$ $(i=1, \cdots, n)$ and

$$\chi_m=1-\left(\frac{1}{p}-\frac{1}{p'}\right)\sum_{i=1}^{m}\frac{1}{r_i}-\frac{1}{p}\sum_{i=m+1}^{n}\frac{1}{r_i}>0. \tag{1}$$

If $f(x_1, \cdots, x_n)$ belongs to $H_p^{(r)}(M_1, \cdots, M_n)$, then (a) as a function of $(x_1, \cdots, x_m)$, for arbitrarily fixed $(x_{m+1}, \cdots, x_n)$, it belongs to $H_{p'}^{(\rho^{(m)})}(M_1^*, \cdots, M_m^*)$, $\rho^{(m)}\equiv(\rho_1^{(m)}, \cdots, \rho_m^{(m)})$, where $\rho_i^{(m)}=r_i\chi_m$, $M_i^*\leqq\alpha\|f\|_p^{(n)}+\beta\sum_{k=1}^{n}M_k$, and α depends only on n, but β depends on n, m, $r_1, \cdots, r_n$; (b) if the condition

$$1-\frac{1}{p}\sum_{i=1}^{n}\frac{1}{r_i}>0 \tag{2}$$

is fulfilled, then for any p, p', m satisfying $1<p\leqq p'\leqq\infty$ and $1\leqq m\leqq n$ there exists a function f belonging to $H_p^{(r)}$ but not to $H_{p'}^{(\gamma)}$, $(\gamma)\equiv(\rho_1^{(m)}, \cdots, \rho_{i-1}^{(m)}, \rho_i^{(m)}+\epsilon, \rho_{i+1}^{(m)}, \cdots, \rho_m^{(m)})$ for any choice of $\epsilon>0$ and of the constants $M_1^*, \cdots, M_m^*$. It is shown in the present article that (b) remains valid if (2) is replaced by the less restrictive condition (1), so that the assertion (a) cannot be strengthened for $p(>1)$, p', m, and n satisfying (1). *M. G. Arsove* (Seattle, Wash.).

Zeller, K. Kondensation von Singularitäten. Math.-Phys. Semesterber. **3**, 207–213 (1953).

Expository paper.

Aruffo, Giulio. Sulle condizioni di validità della formula di Green-Stokes generale. Pont. Acad. Sci. Acta **15**, 1–13 (1951). (Latin summary)

The general Green-Stokes formula is proved under conditions less restrictive than those of de Rham and Kodaira [Harmonic integrals, Inst. Advanced Study, Princeton, 1950; these Rev. **12**, 279]. The argument is elementary and in the plane case it generalizes the well-known form of Cauchy's theorem due to Goursat. No extension on the

lines of Lorentz [Math. Z. 51, 61–81 (1945); these Rev. 9, 232] and Federer [Trans. Amer. Math. Soc. 58, 44–76 (1945); these Rev. 7, 199] is attempted. *L. C. Young.*

Theory of Functions of Complex Variables

Verblunsky, S. On the curvature of level curves. Math. Ann. 125, 472–476 (1953).

It is proved that if $w=w(z)$ is analytic for $|z|\leqq 1$, $w'(z)\neq 0$ on this set and $K(e^{i\theta})\geqq k=\text{const.}$, where $K(z)$ is the curvature of the curve $w=w(re^{i\theta})$ ($r=\text{const.}$), then $K(z)\geqq 2k\{1+[1-k|w'(z)|(1-r^2)]^{\frac{1}{2}}\}^{-1}$ for $|z|<1$; it had previously been proved by Walsh [Proc. Nat. Acad. Sci. U. S. A. 23, 84–89 (1937)] that $K(z)\geqq k|z|^{-1}$ for $|z|<1$. The proof is based on an inequality of Peschl [Math. Ann. 106, 574–594 (1932)] for which a simple new proof is given.

W. Kaplan (Ann Arbor, Mich.).

Stečkin, S. B. Estimate of the remainder of a Taylor series for certain classes of analytic functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 461–472 (1953). (Russian)

Let $B^{(r)}$ denote the class of power series $F(z)$ regular for $|z|<1$ and satisfying $|F^{(r)}(z)|\leqq 1$. The author generalizes several results which were known for $r=0$ or 1. Let $\sigma_n(z, F)$ be the $(C, 1)$ means of the power series and $\rho_n(z, F)=F(z)-\sigma_n(z, F)$. For $r=1$, $|\rho_n|\leqq 3/(n+1)$, while for $r\geqq 2$, $|\rho_n|=zF'(z)/(n+1)+O((n+1)^{-r})$, $n\geqq r-2$, where the O-term is uniform with respect to z and with respect to the whole class $B^{(r)}$. Let $p_n(z, F)$ be the partial sums of the power series, $r_n(z)$ the corresponding remainders. For $n\geqq r-1$, $|r_n(z, F)|=-z^r(n+1)^{-r}p_{n-r}(z, F^{(r)})+O((n+1)^{-r})$, uniformly (as before). Hence, in particular,

$$R_n(F)=\max_{|z|\leqq 1}|r_n(z, F)|=O(n^{-r})$$

if and only if

$$\max|p_n(z, F^{(r)})|=O(1);\quad \max|r_n(z, F)|=o(n^{-r}(\log n)^{\frac{1}{2}}).$$

Let $R_n[B^{(r)}]$ be the supremum of $r_n(z, F)$ for F in $B^{(r)}$. Then

$$R_n[B^{(r)}]=\pi^{-1}(n+1)^{-r}\log(n+1)+O((n+1)^{-r}),\quad n\geqq r-1.$$

The dominant term is exact, and the corresponding generalization of Neder's extension of Bohr's theorem [$r=0$; Neder, Math. Z. 11, 115–123 (1921)] holds.

The author uses some Tauberian theorems on $(C, 1)$ summability which are of independent interest. Let s_n, t_n be the partial sums of (S) $\sum_{k=0}^{\infty}u_k$, (T) $\sum_{k=0}^{\infty}(k+1)u_{k+1}$, and σ_n, τ_n the corresponding $(C, 1)$ partial sums. It is well known [cf. Hardy, Divergent series, Oxford, 1949, p. 131; these Rev. 11, 25] that (S) converges, to s (say) if (T) is summable $(C, 1)$. The author proves, for example, that if (T) is bounded $(C, 1)$ and $t_n=o(n)$ then (S) converges and $s-\sigma_n=O(1/n)$; if $\lambda_n\downarrow 0$, (T) is summable $(C, 1)$ to t and $|t-\tau_n|\leqq\lambda_n$, then (S) converges to s and $s-\sigma_n=t/(n+1)+O(\lambda_n/n)$.

R. P. Boas, Jr. (Evanston, Ill.).

***Valiron, Georges. Sur une construction de H. Poincaré.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 239–243. Casa Editrice Perrella, Roma, 1953.

In the proof of a theorem on functions with lacunary spaces, Poincaré constructed a certain function $F(x)\theta(x)$, continuous on the circle C ($|x|=1$), and analytic on C except at the points $x=\pm 1$; and he asserted that its Fourier series (1) $\sum a_n e^{-in\omega}+\sum b_n e^{in\omega}$ (where $x=e^{i\omega}$) converges on C [Amer. J. Math. 14, 201–221 (1892)]. The author notes that (i) Poincaré failed to establish convergence of (1) on C; (ii) by a theorem of Fejér and M. Riesz, the series (1) converges uniformly on every closed arc of C which passes through neither of the two points $x=\pm 1$, and this suffices for the proof of Poincaré's theorem; (iii) with appropriate precautions in the choice of the factor $\theta(x)$, the series (1) converges absolutely on $|x|=1$.

G. Piranian.

Al'per, S. Ya., and Ivanov, V. V. On the approximation of functions by partial sums of a series of Faber polynomials. Doklady Akad. Nauk SSSR (N.S.) 90, 325–328 (1953). (Russian)

The rapidity of convergence of the expansion of an analytic function $f(z)$ in a series of Faber polynomials $\sum_{n=0}^{\infty}a_n\phi_n(z)$ is related to the modulus of continuity of the function. Typical theorem. Let a region D be bounded by a finite number of smooth, non-tangent arcs making exterior angles one with the other of not less than $\lambda\pi$ ($0<\lambda\leqq 1$). Let $f(z)$ be regular in D and possess a continuous pth derivative in $\bar{D}$. Let $\omega_p(\delta)$ be the modulus of continuity of $f^{(p)}(z)$ in $\bar{D}$. Then, given an arbitrary $\epsilon>0$, there exists for $p>0$ a constant C_1 independent of n and z such that

$$\left|f(z)-\sum_{k=0}^{n}a_k\phi_k(z)\right|<C_1 n^{-\lambda p+\epsilon}\omega_p(n^{-\lambda+\epsilon}),\quad z \text{ in } \bar{D},$$

while for $p=0$, we have

$$\left|f(z)-\sum_{k=0}^{n}a_k\phi_k(z)\right|<C_2\omega(n^{-\lambda+\epsilon})(\log n)^2,\quad z \text{ in } \bar{D}.$$

P. Davis (Washington, D. C.).

Blambert, Maurice. Un problème de composition des singularités des séries de Dirichlet générales. Acta Math. 89, 217–242 (1953).

This paper gives conditions under which the point $\gamma=\alpha+\beta$ where α is a pole of $f(s)=\sum a_n e^{-\lambda_n s}$ and β is a singularity of $\varphi(s)=\sum b_n e^{-\mu_n s}$, is a singularity of the Hadamard-Mandelbrojt composed series $H_M[f(s), \varphi(s)|k]=\sum a_n^{(k)}e^{-\mu_n s}$. Some of the results were announced previously [these Rev. 9, 576; 11, 20; 12, 88; 13, 837] and constitute a generalization of well known results of Borel and Pólya in the case of the Hadamard composition of Taylor series. The precise conditions are too complicated to be reproduced here. There are two groups of theorems. In the first group the main condition is that there is only one pair of singularities, α a pole of $f(s)$ and β a singularity of $\varphi(s)$ so that $\gamma=\alpha+\beta$, and the proof uses differential equations. In the second group one allows a finite number of couples (α_n, β_n), α_n a pole of $f(s)$ and β_n a singularity of $\varphi(s)$, $n=0, 1, \cdots, N$, so that $\gamma=\alpha_n+\beta_n$, $\alpha_n\neq 0$, $\text{Re}\{\alpha_n\}\neq\text{Re}\{\alpha_m\}$, $n\neq m$, and the α_n's don't satisfy any relation of the type $P(\alpha_0, \alpha_1, \cdots, \alpha_N)=0$, where P is a polynomial with integral coefficients. The conclusion in this case is that γ is a singularity of $H_M[f(s), \varphi(s)|k]$ for k sufficiently large, and the proof uses difference-differential equations.

S. Agmon.

Tanaka, Chuji. Note on Dirichlet series. V. On the integral functions defined by Dirichlet series. I. Tôhoku Math. J. (2) 5, 67–78 (1953).

Suppose that the Dirichlet series $\sum_1^{\infty}a_n e^{-\lambda_n s}\equiv F(s)$ converges uniformly in every right half plane to an entire function. The object of the note is to express the order ρ

and the type k of $F(s)$ in terms of $\{a_n\}$ and $\{\lambda_n\}$. A number of inequalities are proved, among which we note:

$$-1/\rho_c \leqq -1/\rho \leqq -1/\rho_u \leqq -1/\rho_c + \limsup_{x\to\infty} (x \log x)^{-1} \log^+ N(x).$$

Here

$$-1/\rho_c = \limsup_{n\to\infty} (\lambda_n \log \lambda_n)^{-1} \log |a_n|,$$

$$\rho_u = \limsup_{\sigma\to-\infty} (-\sigma)^{-1} \log^+ \log^+ M_u(\sigma),$$

$$N(x) = \sum 1, \quad [x] \leqq \lambda_n < x,$$

and $M_u(\sigma)$ is the supremum of the partial sums of the series on the vertical line $\Re(s)=\sigma$. Similar inequalities are found for k. *E. Hille* (New Haven, Conn.).

Sunyer i Balaguer, Ferran. Sur le théorème de Denjoy-Carleman-Ahlfors. C. R. Acad. Sci. Paris **237**, 548–550 (1953).

The number of asymptotic values and their "rotation index" are restricted for integral functions of finite order by hypotheses on the rapidity of convergence towards the values. With convergence "of positive type" the value is defective in the sense of Nevanlinna. *A. J. Macintyre.*

Shah, S. M., and Khanna, Girja. On entire functions of infinite order. Math. Student **21**, 47–48 (1953).

The first author has previously proved that for an entire function $f(z)$ of infinite order

$$(*) \qquad \liminf_{r\to\infty} \frac{\log M(r,f)}{\nu(r,f)} = 0$$

[Quart. J. Math., Oxford Ser. (2) **1**, 112–116 (1950); these Rev. **12**, 16]. Here $M(r,f)$ denotes the maximum modulus of $f(z)$, $\nu(r,f)$ the rank of the maximum term in the power series of $f(z)$. It is shown in the present note that (*) remains valid if $M(r,f)$ is replaced by $rM(r,f')$. *J. Korevaar.*

Singh, S. K. A note on a paper of R. C. Buck. Proc. Indian Acad. Sci. Sect. A. **38**, 120–121 (1953).

Let $f(z)$ be an entire function of finite order ρ and type τ, and let $\limsup r^{-\rho} n(r) = L$, $\liminf r^{-\rho} n(r) = l$, where $n(r)$ is the number of zeros of $f(z)$ in $|z| \leqq r$. The author shows that if $l = \rho\tau$, then $L = l$. [This was also observed by Lakshminarasimhan, J. Indian Math. Soc. (N.S.) **17**, 55–58 (1953); these Rev. **15**, 114]. *R. P. Boas, Jr.* (Evanston, Ill.).

Henriksen, Melvin. On rings of entire functions of finite order. J. Indian Math. Soc. (N.S.) **17**, 59–61 (1953).

Let R be the ring of all entire functions, R^* the subring of those of finite order, and R_λ those of order not exceeding λ. The author has previously shown that R/M is always ring-isomorphic to the complex field, for every maximal ideal M in R [Pacific J. Math. **2**, 179–184 (1952); these Rev. **13**, 954]. He now observes that this is also true for certain types of maximal ideals in the rings R_λ and R^*. These may be characterized by the property that M shall contain at least one function f whose zeros $\{a_n\}$ do not "bunch" in the sense that there exist constants N and h such that there are at most N zeros of f in each of the circles with center a_n, radius $|a_n|^{-h}$. *R. C. Buck* (Madison Wis.).

Davis, Philip. Completeness theorems for sets of differential operators. Duke Math. J. **20**, 345–357 (1953).

In a previous paper [J. Indian Math. Soc. (N.S.) **14**, 139–155 (1950); these Rev. **12**, 713] the author has discussed the solution of the homogeneous linear equations:

$$(1) \qquad \beta_p + \sum_{p+1}^{\infty} a_{pk}\beta_k = 0, \quad p = 0, 1, \cdots.$$

Let $L_n(f) = f^{(n)}(0)/n! + \sum_n^\infty a_{nk} f^{(k)}(0)/k!$ be a sequence of functionals applicable to a class E of analytic functions. A subclass $S \subset E$ is a uniqueness class for $\{L_n\}$ and $\{L_n\}$ is total (complete) over S if $L_n(f) = 0$ for all n implies $f \equiv 0$ for f in S. If $f(z) = \sum_0^\infty c_n z^n$, then $L_n(f) = 0$ becomes $\sum_n^\infty a_{nk} c_k = 0$, which is easily converted into (1). If $A = (a_{nk})$ is any triangular matrix with $a_{nn} = 1$ and $a_{nk} = 0$ for $n > k$, let γ^p_{nm} be the n-by-n determinant formed by selecting from A rows p, $p+1, \cdots, p+n-1$ and columns $p+1, p+2, \cdots, p+n-1$, m. Then, (1) is equivalent to:

$$(-1)^n \beta_p = \sum_{k=n}^{\infty} \gamma^p_{n,\,p+k} \beta_{p+k}, \quad n = 1, 2, \cdots, \quad p = 0, 1, \cdots.$$

The matrix A has property M if numbers M_p exist with $\sum_{k=n}^\infty |\gamma^p_{n,\,p+k}| < M_p$ for all n and p. If $\{p_n\}$ is a positive sequence, let $E(p_n)$ be the class of power series $\sum c_n z^n$ for which $\lim c_n/p_n = 0$. The author proves: A necessary and sufficient condition that $E(P_n)$ be a uniqueness class for $\{L_n\}$ is that the matrix A^* have property M, where $A^* = (a_{nk}p_k/p_n)$. If L_n is specialized to $L_n(f) = f^{(n)}(\alpha_n)/n!$, more detailed results can be achieved. The following theorem is obtained. Let the α_n be subjected to the restriction $\limsup n^{-1} \sum_0^n |\alpha_k| \leqq 1$. Then, a uniqueness class for $\{L_n\}$ is the class of entire functions of exponential type $< e^{-1}$, and this constant cannot be increased. [This throws additional light upon the problem of the exact value of the Whittaker constant. See S. S. Macintyre, J. London Math. Soc. **22**, 305–311 (1948); these Rev. **10**, 27.]

R. C. Buck (Madison, Wis.).

Zmorovič, V. A. On the structure formulas of some classes of univalent functions. Doklady Akad. Nauk SSSR (N.S.) **72**, 833–836 (1950). (Russian)

The author announces a number of theorems on the class of functions $F(z) = z + \cdots$ regular in $|z| < 1$ for which (1) $\Re e^{-i\alpha} F'(z) > 0$ or (2) $\Re e^{-i\alpha} z F'(z)/F(z) > 0$. A generalization to the case $|F(z)| \leqq M$ is indicated.

A. W. Goodman (Lexington, Ky.).

Zmorovič, V. A. On structure formulas of certain classes of analytic functions univalent in a circular ring. Doklady Akad. Nauk SSSR (N.S.) **86**, 465–468 (1952). (Russian)

Single-valued analytic functions $f(z)$, are considered in the annulus $q < |z| < 1$. Such a function is called normed if $\int f dz/z = 2\pi i$, the integral being extended over $|z| = \rho$, $q < \rho < 1$. Theorem 1 gives a Stieltjes integral representation of normed functions with positive real parts. The formula, generalizing the classical Herglotz representation, reads

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(ze^{-i\theta})\,d\mu_1(\theta) + \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(\frac{q}{z}e^{i\theta}\right) d\mu_2(\theta) - 1.$$

Here μ_1, μ_2 are non-decreasing real-valued functions and

$$F(z) = \frac{i}{\pi}\vartheta_1'\left(\frac{\log z}{2\pi i}; q\right) = \frac{1+z}{1-z} + 2\sum_1^\infty \frac{q^{2k}}{1-q^{2k}}(z^k - z^{-k}).$$

Using Theorem 1 the author finds integral representations for functions $f(z)$ satisfying the following conditions: $f(z)$ is univalent and maps circles $|z| = \rho$ onto star-shaped curves (Th. 2), $\operatorname{Re} e^{i\alpha} f'(z) z/f(z) > 0$ for some real α (Th. 3), $f(z)$ maps circles $|z| = \rho$ into convex curves (Th. 4), $f(z)$ has a

simple pole in the annulus and maps the annulus in a one-to-one way onto the complement of two convex bodies (Th. 6). [For similar results in the case of a disc, cf. the paper reviewed above.] *L. Bers* (New York, N. Y.).

Imura, Hideo. A note on bounded functions. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 245–248 (1953).

Let $f(z)=z^p+a_{p+1}z^{p+1}+\cdots$ be regular and $|f(z)|<M$ in $|z|<1$ and let z_1 with $r_1=|z_1|\neq 0$ be its zero nearest $z=0$. The author shows that $f(z)$ is p-valent in $|z|<R$ where R is the smallest positive root of the equation

$$h(\rho)\equiv\rho-rr_1^3(M^2-1)(r_1-r)^{-1}(1-r_1r)^{-1}-r(M^2r_1^2-1)(Mr_1-r)^{-1}(1-Mr_1r)^{-1}=0.$$

If $r_1=1/M$, then $R=N-(N^2-1)^{1/2}$ with

$$N=\tfrac{1}{2}[(M+M^{-1})+\rho^{-1}(M-M^{-1})],$$

which result was obtained by Loomis [Bull. Amer. Math. Soc. **46**, 496–501 (1940); these Rev. **1**, 308]. For $p=1$ the proof consists in showing that $\mathrm{Re}\,(zf'/f)\geqq h(1)>0$ in $|z|<R$ and thus showing also that $f(z)$ is star-shaped with respect to the origin in $|z|<R$. A theorem of Kobori is used in the general case. *M. Marden* (Milwaukee, Wis.).

Kubo, Tadao. Some theorems on bounded analytic functions. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 235–243 (1953).

The following two theorems are proved. Let $f(z)$ be bounded and single-valued in a multiply-connected domain D, and let $f(z)=\sum a_nf_n(z)$ be an expansion of $f(z)$ in terms of a set of functions orthonormalized by means of a boundary norm. (a) If $M(z)=\sum|a_n||f_n(z)|$, then $M(z)=o([d(z)]^{-1})$, where $d(z)$ is the distance of z to the nearest boundary point of D. (b) If $\log(f(z)/z)=g(z)$ is regular and single-valued in D and $\limsup_{z\to t}|g(z)|\leqq\lambda(t)$, where t is on the boundary of D and $\lambda(t)$ is positive and continuous, then $f(z)$ will be univalent in a circle about the origin whose radius can be expressed in terms of a kernel introduced by the reviewer [Trans. Amer. Math. Soc. **69**, 161–178 (1950); these Rev. **12**, 251]. (a) generalizes a theorem of Hardy and (b) one of the reviewer [Amer. J. Math. **71**, 845–852 (1949); these Rev. **11**, 426]. *Z. Nehari* (St. Louis, Mo.).

***Nevanlinna, Rolf. Eindeutige analytische Funktionen.** 2te Aufl. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd XLVI. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. x+379 pp. DM 46.50; Ganzleinen DM 49.50.

The object of this new edition of Nevanlinna's "Eindeutige Analytische Funktionen" is to take account of the developments of the Nevanlinna theory that have appeared since the first edition. The inclusion of such material is selective rather than exhaustive. Emphasis is placed on results which are of methodological importance. The recent work of Lehto [Math. Scand. **1**, 5–17 (1953); these Rev. **15**, 115] and Virtanen's proof of the second fundamental theorem [see the paper reviewed below] are featured. The chapter on plane sets of logarithmic capacity zero has been elaborated. The enlarged bibliography reflects the present state of the theory. References to the more modern developments have also been added to the text. *M. Heins.*

Virtanen, K. I. Eine Bemerkung über die Anwendung hyperbolischer Massbestimmungen in der Wertverteilungslehre der meromorphen Funktionen. Math. Scand. **1**, 153–158 (1953).

Let K_q be the disk $|z|<R$ punctured at the points $z_\nu^{(1)},\cdots,z_\nu^{(q)}$ at which a meromorphic function $w(z)$ in $|z|<R$ assumes given values $a_1,\cdots,a_q$. Extending the method of F. Nevanlinna [Acta Math. **50**, 159–188 (1927)], the author introduces a Poincaré hyperbolic metric $d\sigma(z)$ on K_q and derives an estimate for the characteristic function $T(r)$ in terms of $d\sigma(z)$ and $\bar{N}(r,a_\nu)$; the latter differs from the R. Nevanlinna function $N(r,a_\nu)$ in that each point $z_\nu^{(\nu)}$ occurs in $\bar{N}(r,a_\nu)$ once only. If, in particular, $\Sigma N(r,a_\nu)$ is bounded, then $T(r)$ is at most of order $(q-2)^{-1}\log(R/(R^2-r^2))$. The paper is related to the recent simplified proofs of the principal theorems of R. Nevanlinna [see the book reviewed above]. *L. Sario* (Cambridge, Mass.).

Wittich, Hans. Bemerkung zur Wertverteilung von Exponentialsummen. Arch. Math. **4**, 202–209 (1953).

A variety of remarks on the distribution of values of functions of the form $\sum A_\mu\exp(a_\mu z)$ (finite sums, A_μ constants or polynomials) and of integral-function solutions of differential equations with polynomial coefficients. These remarks are in part complementary to the classical theory and in part explanatory in terms of special examples. *A. J. Macintyre* (Aberdeen).

af Hällström, Gunnar. Kapazitätsbeziehungen bei konformer Abbildung von Einschnittgebieten. Math. Scand. **1**, 131–136 (1953).

The author continues his study [Soc. Sci. Fenn. Comment. Phys.-Math. **16**, no. 13 (1952); Acta Acad. Aboensis **18**, no. 6 (1952); these Rev. **14**, 549, 861] of the conformal mappings of the unit circle onto domains obtained from the unit circle by applying to it an infinity of radial slits ending on the circumference. *Z. Nehari* (St. Louis, Mo.).

Epheser, Helmut, und Stallmann, Friedemann. Konforme Abbildung eines Rechtecks mit Halbkreiskerbe. Z. Angew. Math. Mech. **33**, 268–269 (1953).

***Nevanlinna, Rolf. Uniformisierung.** Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd LXIV. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. x+391 pp. DM 46.50; bound DM 49.50.

Since the appearance of Weyl's celebrated "Idee der Riemannschen Fläche" [Teubner, Leipzig 1913] the theory of abstract Riemann surfaces has experienced a considerable development, especially in the last quarter of a century. In spite of this fact, apart from the monograph of Stoilow, very little has appeared on this subject in book form up to the present. The treatise of Nevanlinna, which is based upon his lectures at Zürich and Helsinki, is the first of a new generation of books to appear on the subject. The objective of the author is to provide an introduction to the theory of abstract Riemann surfaces with uniformization theory being the focal point of interest.

Since Weyl's book a considerable amount of work has been done on the foundations of the subject. Here the fundamental contribution of Radó [Acta Litt. Sci. Szeged **2**, 101–121 (1925)] should be mentioned. In it Radó demonstrated the triangulability of surfaces (with countable base) and also showed that in formulating the notion of a Riemann

surface the countable base condition could be omitted since it appeared as a consequence of the conformal structure. Radó's proof of this fact calls into play the conformal mapping of the universal covering surface of a Riemann surface. With the improvement of methods for solving Dirichlet problems it became apparent that one could dispense with triangulability considerations for treating the conformal mapping of simply-connected Riemann surfaces [cf. M. Heins, Ann. of Math. (2) **50**, 686–690 (1949); these Rev. **11**, 93].

Such tendencies are taken into account in the present work. A noteworthy detail is a direct proof of the Radó theorem that the countable base condition can be omitted which does not call into play the use of covering surfaces but rather is based on the construction of a certain bounded non-constant harmonic function on the given manifold less two disjoint homeomorphs of the unit disc. It is inferred that the region so obtained contains a countable dense subset of points and that it may be metrized with the aid of the auxiliary harmonic function. Thereupon the countable base property is readily inferred.

A number of methods are available for treating existence questions. Of these the author has preferred to follow the alternating procedure of Schwarz and C. Neumann.

In the past two decades considerable attention has been focussed on non-compact ("open") Riemann surfaces. The researches of the author in this direction have played an important role (harmonic measure, extension of the theory of abelian differentials, etc.). The last chapter of the book gives an account of this subject. As the author points out, the development has been so active that it was impossible to take into account the progress in the theory of open Riemann surfaces that occurred during the editorial preparation of the book beyond bibliographical reference.

It should be mentioned that the construction of §2.91, pp. 90–91 has been substantially given by the reviewer [Proc. Amer. Math. Soc. **2**, 951–952 (1951); these Rev. **13**, 547].

The contents follow: I. Algebraic functions. II. The concept of a Riemann surface. III. Function-theoretic foundations. IV. Existence theorems (harmonic functions). V. Closed Riemann surfaces. VI. The Riemann mapping theorem for simply-connected Riemann surfaces. VII. Groups of linear fractional transformations. VIII. Uniformization. IX. Planar (schlichtartig) surfaces. X. Open Riemann surfaces. *M. Heins* (Providence, R. I.).

Sario, Leo. Alternating method on arbitrary Riemann surfaces. Pacific J. Math. **3**, 631–645 (1953).

L'auteur montre comment la méthode alternée de Schwarz peut être appliquée à des surfaces de Riemann quelconques. Il en fait une application à la démonstration de résultats qu'il a publiés dans trois notes [C. R. Acad. Sci. Paris **229**, 1293–1295 (1949); **230**, 42–44, 168–170 (1950); ces Rev. **11**, 342], et qui peuvent être aussi démontrés par sa méthode de l'opérateur linéaire [Trans. Amer. Math. Soc. **72**, 281–295 (1952); ces Rev. **13**, 735]. Citons les principaux résultats, dont certains ont été démontrés autrement par divers auteurs.

Soit R une surface de Riemann quelconque, G un domaine de R dont la frontière A relative à R est formée d'un nombre fini de courbes analytiques closes. Soit u une fonction définie dans $R-G$, s'annulant sur A, harmonique, uniforme et non constante au voisinage de A, et ayant par ailleurs des singularités quelconques.

I. Si G est compact, la condition

$$(1)\qquad \int_A \frac{\partial u}{\partial n} ds = 0$$

entraîne l'existence d'une fonction f sur R telle que 1) f est harmonique, uniforme, non constante sauf là où u présente des singularités, 2) $f-u$ est harmonique bornée (classe HB) dans $R-G$, et a une intégrale de Dirichlet finie sur $R-G$ (classe HD). Si G n'est pas compact, f existe et vérifie ces deux conditions indépendamment de (1); mais si (1) est satisfaite, f est HB et HD sur G.

II. Pour qu'il existe une fonction de Green sur R, il faut et suffit que la frontière idéale de R ait une mesure harmonique positive.

III. Pour qu'il existe une fonction HB ou HD sur R, il faut et suffit qu'il existe un domaine G et une fonction u comme ci-dessus vérifiant (1) et qui soit de classe HB ou HD.

IV. (Réponse à un problème posé par Myrberg en 1948 à l'Université d'Helsinki.) Supposant R de genre positif, il existe sur R des intégrales abéliennes des trois espèces, qui sont de classe HD si on retranche un voisinage des singularités.

V. R étant du type parabolique, soient A et B deux domaines de R non compacts tels que $A \cap B$ soit doublement connexe et bordée par deux courbes de Jordan closes analytiques, et a, b deux fonctions harmoniques uniformes dans $A \cap B$. Si la différence des conjuguées de a et b est uniforme, il existe une fonction harmonique f sur R telle que $f-a$ est harmoniquement prolongeable dans A, $f-b$ dans B, en restant uniformes et bornées. *R. de Possel.*

***Sario, Leo. Construction of functions with prescribed properties on Riemann surfaces.** Contributions to the theory of Riemann surfaces, pp. 63–76. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

Après avoir passé rapidement en revue les différentes méthodes qui ont été utilisées pour démontrer l'existence de fonctions ayant certaines propriétés sur une surface de Riemann, l'auteur résume deux d'entre elles qu'il a introduites lui-même: la méthode de l'opérateur linéaire [Trans. Amer. Math. Soc. **72**, 281–295 (1952); ces Rev. **13**, 735], et la méthode extrémale [ibid. **73**, 459–470 (1952); ces Rev. **14**, 863]. Il précise les résultats essentiels qu'elles lui ont permis d'obtenir et en résume les démonstrations. Il rappelle ensuite les définitions qu'il a introduites de la capacité de la frontière et de celle d'une composante frontière [Bull. Amer. Math. Soc. **58**, 498 (1952)].

Il fait enfin un historique du problème de la classification, et indique les principaux résultats qu'il a obtenus dans cette voie: identité des surfaces à "frontière nulle" et des surfaces sans fonction de Green [voir l'oeuvre analysé ci-dessus]; type de la surface uniformisante de Schottky; introduction des surfaces de "caractère clos" et des surfaces avec "frontière enlevable" [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **50** (1948); ces Rev. **10**, 365]; il cite en particulier les résultats d'Ahlfors [Comment. Math. Helv. **24**, 100–134 (1950); ces Rev. **12**, 90; **13**, 1138].

Il résume alors au moyen de notations condensées tous les résultats connus. La classification est basée: 1) sur l'existence de fonctions harmoniques ou analytiques présentant certains caractères de périodicité, d'univalence, ayant un développement de forme donnée dans un cercle paramétrique; ces fonctions pouvant être bornées, positives, pouvant avoir une intégrale de Dirichlet finie, pouvant omettre un en-

semble de valeurs d'aire positive, pouvant avoir une intégrale à la frontière idéale $\int p d\bar{p}$ négative ($\bar{p}$ = conjuguée de p); 2) sur d'autres propriétés de la surface: son frontière peut être "enlevable", son P-écart ou son Q-écart peuvent être nuls [notions définies dans le premier oeuvre cité], la capacité de la frontière idéale peut être nulle, il peut en être de même de la capacité de chacune de ses composantes frontières. *R. de Possel* (Alger).

Morse, M., and Jenkins, J. The existence of pseudoconjugates on Riemann surfaces. Fund. Math. **39** (1952), 269–287 (1953).

The authors give a new proof of a theorem of Boothby [Amer. J. Math. **73**, 512–538 (1951); these Rev. **13**, 266]: Every regular curve-family filling the plane except for isolated singularities of saddle-point type is topologically equivalent to the family of level curves of a harmonic function. Boothby's proof relied on the theorem that, when there are no singular points, every such family is topologically equivalent to the family of solutions of differential equations $\dot{x}=f(x,y)$, $\dot{y}=g(x,y)$, where f and g have continuous derivatives; a proof of this was outlined by the reviewer [Lectures in topology, Univ. of Michigan Press, 1941, pp. 299–301; these Rev. **3**, 135]. The present paper does not use this theorem and achieves its goal by directly constructing a "pseudo-conjugate" for a "pseudo-harmonic" function having the given family as level curves.

W. Kaplan (Ann Arbor, Mich.).

***Jenkins, James A., and Morse, Marston. Topological methods on Riemann surfaces. Pseudoharmonic functions.** Contributions to the theory of Riemann surfaces, pp. 111–139. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

By a pseudoharmonic function on a Riemann surface G^* is meant a function U which in the neighborhood of each point of U is the composition of a harmonic function and a homeomorphism. By topological analysis the authors prove that any set F of level lines in the complex plane which are locally the level lines of a pseudoharmonic function is in fact the set of level lines of a pseudoharmonic function defined on the whole plane. Much of the topological analysis has been carried out with a view towards the extension of this theorem to multiply-connected Riemann surfaces.

H. L. Royden (Stanford, Calif.).

Aikawa, Sanzō. On extension of Schwarz's theorem. Kōdai Math. Sem. Rep. **1952**, 104–106 (1952).

On trouve dans le livre de Bochner et Martin [Several complex variables, Princeton, 1948; ces Rev. **10**, 366] une généralisation du classique lemme de Schwarz au cas d'un système de n fonctions $f_i(z_1, \cdots, z_k)$ holomorphes dans la boule $\|z\| = (\sum_{1\leq j\leq k} |z_j|^2)^{1/2} \leq 1$, nulles à l'origine. Si en outre les dérivées partielles des f_i s'annulent à l'origine jusqu'à l'ordre m inclus, on a, pour tout $p>1$,

$$\|f(z)\|_p \leq \|z\|^{m+1} \sup_{\|w\|=1} \|f(w)\|_p,$$

où $\|f\|_p$ désigne $(\sum_{1\leq i\leq n} |f_i|^p)^{1/p}$. Tel est le principal résultat de cette note, qui en donne deux applications.

H. Cartan (Paris).

Bochner, Salomon. The theorem of Morera in several variables. Ann. Mat. Pura Appl. (4) **34**, 27–39 (1953).

Plusieurs extensions du théorème de Morera dans différentes directions sont données. 1) Si $f(z_i, \bar{z}_j)$ est continue $(1\leq i\leq k, 1\leq j\leq k)$ dans un domaine D et si

$$\int_{B_{p+q}} \phi = \int_{B_{p+q}} f dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_p} \wedge d\bar{z}_{\beta_1} \wedge \cdots \wedge d\bar{z}_{\beta_q} = 0,$$

pour tout cycle B_{p+q} à $p+q$ dimensions réelles

$$(1\leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq n;\quad 1\leq \beta_1 < \cdots < \beta_q \leq n;\quad 1\leq p+q\leq 2k-1),$$

alors on a $\partial f/\partial z_\alpha = 0$ pour tout indice α différent de $\alpha_1, \cdots, \alpha_p$, et $\partial f/\partial z_\beta = 0$ pour tout indice β différent de $\beta_1, \cdots, \beta_q$; f est ainsi holomorphe de certains z_β, "antiholomorphe" de certains z_α. 2) Si $\int_{B_k} f dz_1 \wedge \cdots \wedge dz_k = 0$ pour tout cycle $B_k = b_1^1 \times b_1^2 \times \cdots \times b_1^k$ produit de cycles dans les plans coordonnés ($b_1^1 = E[z_1 \in b_1^1$; z_j constants pour $j\neq 1]$) et si f est k fois dérivable, alors $\partial^k f/\partial z_1 \cdots \partial z_k = 0$. 3) Si pour un nombre $l\geq 1$ les expressions $\partial^l f/\partial z_{\alpha_1} \cdots \partial z_{\alpha_l}$ sont analytiques (au sens de analytique réel) pour toutes les combinaisons possibles $(\alpha_1, \cdots, \alpha_l)$, f étant $l+1$ fois dérivable, alors f est analytique. 4) Si f est continue et si $\int_{B_k'} f dz_1 \wedge \cdots \wedge dz_k = 0$ pour tout cycle B_k' résultant d'un cycle B_k précédent par une transformation analytique complexe, alors f est holomorphe. *P. Lelong* (Lille).

Aruffo, Giulio. Forme differenziali esterne di classe 0 e funzioni di più variabili complesse. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 381–385 (1953).

This note proves the converse of Bochner's result in the paper reviewed above which states that, if f is a continuous function in a cell C_{2n} of n-dimensional complex affine space, then a necessary condition for

$$\int_{B_{n+q}} f dz^{\alpha_1} \cdots dz^{\alpha_p} d\bar{z}^{\beta_1} \cdots d\bar{z}^{\beta_q} = 0$$

for all cycles B_{p+q} in C_{2n} is that f be analytic in z^β $(\beta\neq\beta_1, \cdots, \beta_q)$ and antianalytic in z^α (i.e., analytic in $\bar{z}^\alpha$) $(\alpha\neq\alpha_1, \cdots, \alpha_p)$. The proof is based on the results of the author's notes [same Atti (8) **13**, 367–372 (1952); **14**, 13–18 (1953); these Rev. **15**, 253], and these methods are also used to establish the direct result.

W. V. D. Hodge (Cambridge, England).

Thimm, Walter. Über ausgeartete meromorphe Abbildungen. I. Über die Änderung der Monodromiegruppe parameterabhängiger analytischer Mannigfaltigkeiten. Math. Ann. **125**, 145–164 (1952).

Thimm, Walter. Über ausgeartete meromorphe Abbildungen. II. Math. Ann. **125** (1952), 264–283 (1953).

Le but essentiel de ces deux articles est de prouver un théorème de dépendance algébrique entre fonctions méromorphes (Hauptsatz II, p. 282), que nous allons expliquer. Soit, dans l'espace C^n de n variables complexes, un sous-ensemble analytique Δ_m passant par l'origine 0 et irréductible en 0, de dimension complexe m. Une "fonction méromorphe" sur Δ_m, au voisinage de 0, est induite par une fonction $p(x)/q(x)$, p et q holomorphes au voisinage de 0 dans C^n, $q(x)$ non identiquement nulle sur Δ_m; elle est identiquement nulle si $p(x)=0$ identiquement sur Δ_m. Des fonctions méromorphes $t_1, \cdots, t_k$ sur Δ_m sont "analytiquement liées" s'il existe $\lambda_1, \cdots, \lambda_k$ holomorphes, non toutes identiquement nulles, telles que $\sum_i \lambda_i dt_i = 0$ sur Δ_m au voisinage de 0; elles sont "algébriquement liées" s'il existe un polynôme $Q\neq 0$ tel que $Q(t_1, \cdots, t_k)$ soit une fonction méromorphe identiquement nulle sur Δ_m.

Considérons k fonctions méromorphes $t_i=p_i/q_i$ sur Δ_m; soit Δ_m' l'ensemble des points $x \in \Delta_m$ où, pour tout i, $p_i(x)$ et $q_i(x)$ ne soient pas nulles toutes deux; alors $t_1, \cdots, t_k$ définissent une application analytique de Δ_m' dans le produit P^k de k droites projectives complexes. Soit $L \subset P^k$ l'ensemble des limites des points $(t_1(x), \cdots, t_k(x))$ quand $x \in \Delta_m'$ tend vers 0. L'ensemble L est un ensemble algébrique (i.e., défini par un nombre fini d'équations algébriques dans P^k); cf. un article ultérieur de Thimm [Math. Z. 57, 456–480 (1953); ces Rev. 14, 971]; cela résulte d'ailleurs facilement d'un théorème classique de Chow sur les sous-ensembles analytiques de l'espace projectif complexe [Amer. J. Math. 71, 893–914 (1949); ces Rev. 11, 389]. Faisons maintenant les hypothèses suivantes: (a) le sous-ensemble $L \subset P^k$ possède un point intérieur (donc $L=P^k$); (b) on considère une fonction u méromorphe sur Δ_m, telle que $u, t_1, \cdots, t_k$ soient analytiquement liées sur Δ_m. Alors, sous les hypothèses (a) et (b), les fonctions $u, t_1, \cdots, t_k$ sont algébriquement liées.

Ce théorème n'est pas énoncé exactement sous cette forme: dans son Hauptsatz II, l'auteur fait, outre l'hypothèse (b), des hypothèses en apparence plus restrictives que (a). La démonstration du théorème occupe toute la partie II du travail; on pourrait d'ailleurs la simplifier en utilisant le théorème de Chow.

Quant à la partie I, de lecture difficile, elle a pour but d'aboutir à un théorème 9 dont on a besoin dans II, mais qui malheureusement n'est nulle part énoncé (ni démontré) explicitement. Cependant, rien que dans la partie I, l'énoncé des définitions et des théorèmes ne tient pas moins de 250 lignes du texte, sans compter les lemmes et les corollaires. Dans ce labyrinthe, voici ce que le rapporteur croit avoir compris: Soit M un sous-ensemble analytique irréductible, de dimension m, dans le produit $X \times T$ de deux variétés analytiques complexes (T étant de dimension k), et soit f: $M \to T$ l'application induite par la projection $X \times T \to T$. Alors, pour les points t du complémentaire d'un sous-ensemble maigre [au sens de Bourbaki, Topologie générale, Ch. IX, Hermann, Paris, 1948, p. 74; ces Rev. 10, 260] de T, $f^{-1}(t)$ est vide ou ses composantes irréductibles sont toutes de dimension $m-k$; de plus, pour chaque t du complémentaire d'un ensemble maigre de T, on a la propriété suivante: soit $x \in f^{-1}(t)$, et soit N un des germes irréductibles de M au point x; alors $N \cap f^{-1}(t)$ est irréductible comme sous-ensemble analytique de $X \times \{t\}$.

Ce résultat est obtenu par l'étude du "groupe de monodromie" d'une équation algébrique $P(z; x_1, \cdots, x_n)=0$, où le polynôme $P=z^k+\lambda_1 z^{k-1}+\cdots+\lambda_k$ a ses coefficients λ_i holomorphes en $x_1, \cdots, x_n$, le discriminant de P n'étant pas identiquement nul. Pour chaque point $x_i=a_i$, on a un groupe de monodromie $G(a_1, \cdots, a_n)$, qui n'est autre que le groupe de Galois de l'équation $P(z)=0$ à coefficients dans le corps des germes de fonctions méromorphes au point (a_i). Donnons des valeurs constantes $t_{p+1}, \cdots, t_n$ aux variables $x_{p+1}, \cdots, x_n$; par cette spécialisation, P devient un polynôme $P_t(z; x_1, \cdots, x_p)$ à coefficients holomorphes en $x_1, \cdots, x_p$; soit $G_t(a_1, \cdots, a_p)$ son groupe de monodromie au point $a_1, \cdots, a_p$. Le résultat principal est le suivant: $a_1, \cdots, a_p$ étant donnés, il existe un ensemble ouvert partout dense de points $(t_{p+1}, \cdots, t_n)$ tels que

$$G_t(a_1, \cdots, a_p)=G(a_1, \cdots, a_p, t_{p+1}, \cdots, t_n).$$

En particulier, si $P(z)$ est irréductible dans le corps des germes méromorphes en $x_1, \cdots, x_n$ au point

$$(a_1, \cdots, a_p, t_{p+1}, \cdots, t_n),$$

le polynôme $P_t(z)$ obtenu par spécialisation est irréductible dans le corps des germes méromorphes en $x_1, \cdots, x_p$ au point $(a_1, \cdots, a_p)$. *H. Cartan* (Paris).

Tillmann, Heinz-Günther. Randverteilungen analytischer Funktionen und Distributionen. Math. Z. 59, 61–83 (1953).

This paper is closely related to the work of G. Köthe in two previously published papers: one on the subject of duality in function theory [J. Reine Angew. Math. 191, 30–49 (1953); these Rev. 15, 132], and one in which Köthe introduced the concept of a 'boundary distribution' [Math. Z. 57, 13–33 (1952); these Rev. 14, 563]. The present paper deals with functions of n complex variables, whereas Köthe's work was all on spaces whose elements are functions of a single complex variable. In most respects Tillmann's work is a direct generalization of that of Köthe, the principal generalization being that of going from one to several complex variables. The topics include the following: a duality theorem for a certain locally convex linear topological space of analytic functions and the corresponding space of continuous linear functionals; an explanation of the reason for calling one of these latter linear functionals a 'boundary distribution', and a consideration of relationships between such boundary distributions and the theory of distributions originated by L. Schwartz. *A. E. Taylor.*

Vekua, I. N. The general representation of functions of two independent variables admitting a derivative in the sense of S. L. Sobolev and the problem of primitives. Doklady Akad. Nauk SSSR (N.S.) 89, 773–775 (1953). (Russian)

Let T be a bounded domain of the z-plane, the boundary of which consists of $(m+1)$ rectifiable Jordan curves. If $u(z)$ and $\omega(z)$ are L_1 in T, and if for every continuously differentiable $v(z)$ which vanishes outside a compact subset of T

$$\iint_T \{\omega(z)v(z)+u(z)\partial v(z)/\partial \bar{z}\}dxdy=0,$$

the author writes: $u \in S$, $\omega=\partial u/\partial \bar{z}$. The following theorems are proved. I. If $u \in S$, $\partial u/\partial \bar{z}=0$, u is analytic. II. Set

$$f(z)=u(z)+\frac{1}{\pi}\iint_T \frac{\omega(\zeta)d\xi dn}{\zeta-z}.$$

If $u \in S$, $\omega=\partial u/\partial \bar{z}$, then f is L_1 and analytic. If ω and f are L_1 and f is analytic, then $u \in S$ and $\omega=\partial u/\partial \bar{z}$. III. If $u \in S$ and $\partial u/\partial \bar{z}$ is L_p, $p>2$, then u is continuous on the closure of T. IV. If $u \in S$, u is continuous on the closure of T, and $\partial u/\partial \bar{z}$ is L_p, $p>1$, then $\int u(z)dz=2i\iint_T \partial u/\partial \bar{z}\, dxdy$, the line integral being taken over the boundary. *L. Bers.*

***Bers, Lipman. Theory of pseudo-analytic functions.** Institute for Mathematics and Mechanics, New York University, New York, 1953. i+iii+187 pp. (mimeographed)

The theory of pseudo-analytic functions was first announced by the author in two notes [Proc. Nat. Acad. Sci. U. S. A. 36, 130–136 (1950); 37, 42–47 (1951); these Rev. 12, 173; 13, 352]. These lecture notes not only contain proofs and extensions of the results previously announced but give a self-contained and comprehensive treatment of the subject.

The author sets as his goal the development of a function theory for solutions of linear, elliptic, second order partial

differential equations in two independent variables (or systems of two first-order equations). One of the chief stumbling blocks in such a task is the fact that the notion of derivative is a hereditary property for analytic functions while this is clearly not the case for solutions of general second order elliptic equations.

We write functions of x, y as functions of z without implying analyticity. Let $F(z)$, $G(z)$ be defined in a domain D in the $z=x+iy$ plane. Assume (i) that $\operatorname{Im}(\bar{F}G)>0$ in D and (ii) that F and G have Hölder-continuous first derivatives with respect to x and y. If $w(z)$ is any function defined in D then condition (i) implies that for each point $z_0 \in D$ there are real constants λ_0, μ_0 such that $w(z_0)=\lambda_0 F(z_0)+\mu_0 F(z_0)$. The author defines the (F, G)-derivative of w (written $\dot{w}$) at z_0 as

$$\dot{w}(z_0)=\lim_{z\to z_0}\frac{w(z)-\lambda_0 F(z)-\mu_0 G(z)}{z-z_0}$$

A function w is termed (F, G)-pseudo-analytic (or simply pseudo-analytic) in a domain D if $\dot{w}$ exists at each point of D. The function (F, G) is called a generating pair which for the case of analytic functions consists of the constants $(1, i)$. It is a remarkable fact that if $\dot{w}$ exists in a domain D then, for $\bar{D}_1 \subset D$, $\dot{w}$ itself possesses such a derivative with respect to another pair (F_1, G_1) also having properties (i) and (ii). It is shown that pseudo-analytic functions satisfy the system of equations: $w_{\bar{z}}=aw+b\bar{w}$ $(2\partial/\partial\bar{z}=\partial/\partial x+i\partial/\partial y)$ which for the special case $a=b=0$ becomes the Cauchy-Riemann equations. The functions a and b are easily expressible in terms of F, G and their first derivatives.

It is natural to seek relations between analytic and pseudo-analytic functions. Many such are clearly brought out by the similarity principle which is an important tool in the development of the theory. Two functions w and f defined in a domain D are termed similar if there exists a function $s(z)$ continuous and different from zero in $\bar{D}$ such that $w=sf$ in D. The similarity principle states that under very general conditions every pseudo-analytic function defined in D is similar to an analytic function and conversely. It is easy to see the importance of such a principle in establishing theorems concerning zeros, poles, topological behavior, etc. of pseudo-analytic functions.

Chapter 1 begins with the definition of (F, G) derivative. Then Carleman's theorem on the isolated character of zeros of second-order elliptic equations is proved. (A new, simple proof of a stronger form of this theorem is given in appendix III.) The final section discusses certain geometrical properties of pseudo-analytic functions. In order to discuss these the author associates with every pseudo-analytic function w a "pseudo-analytic function of the second kind" $\omega(z)=\varphi+i\psi$. The real-valued functions φ and ψ are determined by the equation $w=\varphi F+\psi G$. It turns out that pseudo-analytic functions of the second kind are interior in the sense of Stoilov and quasi-conformal.

The second chapter begins with a discussion of the properties of generating pairs. For each pair (F, G) one can find a pair (F_1, G_1) corresponding to the derivative $\dot{w}$. This may be continued, yielding a "generating sequence", and the question of the periodicity of this sequence is discussed. When does it happen that after a finite number of derivatives (say n) we have (F_n, G_n) equivalent to (F, G)? (Two pairs are equivalent if they give rise to the same system: $w_{\bar{z}}=aw+b\bar{w}$.) Several theorems are proved although the general problem remains unsolved. (F, G)-integration is introduced and the analogues of Cauchy's and Morera's theorems are proved. Then it is shown that the standard convergence theorems for analytic functions continue to hold. The last section is devoted to a discussion of pseudo-analytic functions on Riemann surfaces.

Consider the linear elliptic equation

$$(*)\qquad Au_{xx}+2Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0$$

or the elliptic system

$$(**)\qquad \begin{aligned} u_x&=A_1V_x+B_1v_y+C_1u+D_1v\\ u_y&=A_2V_x+B_2v_y+C_2u+D_2v. \end{aligned}$$

Suppose that $A, B, \cdots, F$ are Hölder-continuous and A, B, C have Hölder-continuous first derivatives (or the analogous conditions in (**)). In chapter 3 it is shown that the theory of second-order elliptic equations (*) (or of elliptic systems (**)) is identical with the theory of pseudo-analytic functions. The chapter begins with a proof of the main existence theorem necessary to show that solutions of arbitrary elliptic equations can actually be expressed by pseudo-analytic functions. A detailed analysis of certain classes of generating pairs is then given and the similarity principle is proven in its most general form.

A pseudo-analytic function which in the extended plane has only poles is called rational. The rational pseudo-analytic function w (with respect to a given (F, G) of course) which satisfies the relation $\lim_{z\to z_0}[w(z)/\alpha(z-z_0)^n]=1$ and has no other zeros or poles except at $z=\infty$ is termed a global formal power of degree n (n, an integer) and denoted $Z^{(n)}(\alpha, z_0, z)$. These powers are related to pseudo-analytic functions as ordinary powers are to analytic functions. Chapter 4 discusses the main properties of these functions. Also discussed are general rational functions, interpolation by "polynomials", and differentiation of powers. In addition there is a section on the existence of local formal powers.

Chapter 5 begins with a proof of Cauchy's formula which takes a form completely analogous to that for analytic functions. If D is a domain with smooth boundary C then

$$w(z)=-\int_C Z^{(-1)}\left[\frac{w(t)dt}{2\pi i}, t; z\right]$$

for $z \in D$, and the integral on the right vanishes for z not in D. The remaining sections are devoted to formal power series of global powers. Analogues of Taylor and Laurent series are given; the coefficients of the Taylor series are shown to satisfy "Cauchy inequalities". Runge's theorem to the effect that a pseudo-analytic function may be approximated by rational pseudo-analytic functions is proven. The power of Cauchy's formula in the above form is exhibited in this proof which follows the lines of the original proof for analytic functions.

The sixth and final chapter discusses multiple-valued pseudo-analytic functions: logarithm, global powers with fractional exponents, algebraic functions, etc. With the aid of the logarithm the existence of the fundamental solution in the large for a linear, second order, elliptic equation is established. The concluding section gives a brief discussion of uniformization.

While the six chapters consist entirely of new material (except for Carleman's theorem and some potential-theoretical lemmas), the book with the aid of two appendices is self-contained. Appendix I contains a development of the Riesz theory for the Fredholm alternative (used in proving the existence theorem of chapter 3). Appendix II gives an account of the first boundary-value problem for a second order linear homogeneous equation of elliptic type. This is used in chapters 1 and 3. In this appendix the author gives

a new, simple proof of Privaloff's theorem on conjugate functions.

All footnotes, explanatory remarks, and references to the literature are gathered as Notes in appendix III. Notes 1 and 2 discuss the history of the problem and contributions of other authors. Note 21 gives a proof due to Agmon and Bers [see the paper reviewed second below] that the natural domain of convergence of the Taylor series for a pseudo-analytic function is a circle with at least one singularity on the boundary. Note 17 states that the theory of pseudo-analytic functions has been applied to non-linear equations, in particular to the equation of gas dynamics. It will be interesting to see what further results ensue from this far reaching development. *M. H. Protter.*

Kriszten, Adolf. Hyperkomplexe und pseudo-analytische Funktionen. Comment. Math. Helv. **26**, 6–35 (1952).

The author's aim seems to be a unification of various generalizations of complex function-theory by means of an appropriate formalism, in particular by using the differential forms of Cartan. The first part deals with hypercomplex analytic functions. Let A be an associative algebra over the reals, $c_0=1, c_2, \cdots, c_N$ a basis of A, L_z and L_w subspaces of A generated by $(c_0, c_1, \cdots, c_{n-1})$, $n \leqq N+1$, and $(c_{h_1}, \cdots, c_{h_m})$, $m \leqq N+1$, respectively. One sets

$$z=\sum c_h x_h, \quad w=\sum c_{h_k} u_k(x_0, \cdots, x_{n-1}), \quad w^{(i)}=\frac{\partial w}{\partial x_i}=\sum c_{h_k} u_k^{(i)}$$

and considers w as a function of z. Then $w(z)$ is called S-analytic [S stands for Scheffers, cf. G. Scheffers, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Cl. **45**, 828–848; **46**, 120–134 (1894)] from the right (left) if $d(w \wedge dz)=0$ ($d(dz \wedge w)=0$). Such functions always exist if the elements of L_z commute; there exist derivatives $(w^{(0)})$ which are themselves S-analytic, and primitive functions.

On the other hand, if $\delta(w \wedge dz)=0$ (or $\delta(dz \wedge w)=0$, δ being the Euclidean codifferential), $w(z)$ is called F-analytic from the right (left). F stands for Fueter who gave this definition for quaternions. The author discusses the integral theorems resulting from these definitions, and the connection of S-analyticity and F-analyticity with the harmonicity definitions for differential forms given by Hodge and by de Rahm.

The second part deals with pseudo-analytic functions $w=u(x, y)+iv(x, y)$, u and v being connected by first-order partial differential equations with variable coefficients. The author reports on some results by Bers and Gelbart, Bergman, Položiĭ, Bers, and Lukomskaya and shows how these results can be stated and derived using (2×2) matrices instead of complex numbers. [The reviewer believes that the author's assertion on the advantages of matrix-formalism is refuted by the general theory of pseudo-analytic functions [L. Bers, Proc. Nat. Acad. Sci. U. S. A. **37**, 42–47 (1951); these Rev. **13**, 352; see also the preceding review; and I. N. Vekua, Mat. Sbornik N.S. **31** (**73**), 217–314 (1952); these Rev. **15**, 230]. But this theory was not available to the author at the time when the paper was written.]

L. Bers (New York, N. Y.).

Agmon, Shmuel, and Bers, Lipman. The expansion theorem for pseudo-analytic functions. Proc. Amer. Math. Soc. **3**, 757–764 (1952).

Let $F(z)$, $G(z)$ be two complex-valued functions defined in a domain Ω of the $z=x+iy$ plane. (F, G) is called a generating pair if the partial derivatives F_x, G_x, F_y, G_y exist and satisfy a Hölder condition, and if $\operatorname{Im} \bar{F}G>0$. A function $w(z)$ defined in a subdomain $D \subset \Omega$ admits the unique representation

$$w(z)=\phi(z)F(z)+\psi(z)G(z)$$

with real ϕ and ψ. It is called regular pseudo-analytic in D if at every point z_0 of D the (F, G)-derivative

$$\dot{w}(z_0)=\lim_{z \to z_0}\left[\frac{w(z)-\phi(z_0)F(z)-\psi(z_0)G(z)}{z-z_0}\right]$$

exists and is finite. The theory of pseudo-analytic functions was formulated earlier [Bers, Proc. Nat. Acad. Sci. U. S. A. **36**, 130–136 (1950); **37**, 42–47 (1951); these Rev. **12**, 173; **13**, 352; see also the second preceding review]. Classical function theory is the special case $F=1$, $G=i$. The present note settles a question left open in the original formulation of the theory, namely, the exact analogues of the Taylor and Laurent expansions for pseudo-analytic functions. The Taylor expansion theorem proved is the following: If $w(z)$ is a regular pseudo-analytic function defined for $|z-z_0|<R$, then w admits a unique expansion of the form

$$w(z)=\sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z)$$

which converges for $|z-z_0|<R$. This theorem holds for normal generating pairs (F, G). The "global" formal powers appearing in the Taylor expansion are the pseudo-analytic functions which are "similar" to the ordinary powers $a(z-z_0)^n$. The essential difficulty lies in showing that the Taylor series expansion converges throughout $|z-z_0|<R$, and not just in a smaller concentric circle. Expansion theorems obtained earlier [Bers, Amer. J. Math. **72**, 705–712 (1950); these Rev. **13**, 352; Bers and Gelbart, Trans. Amer. Math. Soc. **56**, 67–93 (1944); these Rev. **6**, 86] for "local" formal powers are independent of the present results.

J. B. Diaz (College Park, Md.).

Rizza, Giovanni Battista. Sulle funzioni analitiche nelle algebre ipercomplesse. Pont. Acad. Sci. Comment. **14**, 169–194 (1950). (Latin summary)

Soit A une algèbre associative de rang $n+1$ sur le corps réel; on l'identifie à R^{n+1} en posant $x=(x_0, \cdots, x_n)$ pour chaque $x=\sum_0^n x_h i_h \varepsilon A$, où $\{i_h\}$ est une base de A. L'auteur considère des fonctions w de x $(w=\sum_0^n w_h i_h)$ définies dans des domaines Ω de R^{n+1}, en supposant toujours que les w_h sont des fonctions continûment dérivables des x_h dans Ω.

Il y a plusieurs manières de généraliser à ces fonctions le concept d'analyticité. L'auteur développe d'abord le point de vue de Fueter [Elemente der Math. **3**, 89–94 (1948); ces Rev. **10**, 111]. En posant

$$Dw=\sum_0^n \frac{\partial w}{\partial x_h} i_h, \quad D^*w=\sum_0^n i_h \frac{\partial w}{\partial x_h},$$

on dit que $w(x)$ est analytique à droite [à gauche] dans Ω, si l'on a $Dw=0$ [$D^*w=0$] dans Ω. A l'aide du calcul différentiel extérieur de Cartan, l'auteur établit une formule générale [trop compliquée pour être rapportée ici] qui résume $2^{n+1}-1$ théorèmes intégraux portant sur variétés d'intégration dont la dimension varie de 0 à n; en particulier, dans le cas de dimension n, on retrouve un résultat de Fueter. De cette formule on déduit un autre groupe de théorèmes intégraux, qui peuvent être envisagés comme la généralisation des théorèmes de Cauchy, de Morera et de Severi.

Un deuxième point de vue est considéré, en supposant en plus que l'algèbre A est commutative et munie d'unité.

L'auteur dit alors que $w(x)$ est monogène dans Ω, quand, pour chaque $x \in \Omega$, il existe un élément $w'(x)$ de A qui est la limite de $\Delta w/\Delta x$ $[\Delta x = \bar{x} - x, \Delta w = w(\bar{x}) - w(x)]$ pour $\bar{x} \to x$ sur chaque ligne régulière passant par x, pourvu que Δx soit inversible (définition équivalente à celle de "fonction totalement dérivable" de Spampinato). Une condition nécessaire et suffisante pour que $w(x)$ soit monogène dans Ω est que $\int_\Gamma w dx = 0$ pour tout 1-cycle Γ homologue à zéro dans Ω (généralisation des théorèmes de Cauchy et de Morera). Le problème de la généralisation de la formule de Cauchy semble plus difficile; l'auteur étudie le problème dans le cas particulier de l'algèbre de Sobrero en obtenant des résultats plus généraux que ceux de Sobrero.

Note du reviewer: Il serait intéressant d'étudier les rapports entre ces résultats et ceux de E. Lorch dans sa théorie des fonctions analytiques dans des algèbres de Banach commutatives complexes [Trans. Amer. Math. Soc. **54**, 414–425 (1943); ces Rev. **5**, 100]. *J. Sebastião e Silva.*

Fedorov, V. S. On monogeneity of hypercomplex functions. Mat. Sbornik N.S. **32(74)**, 249–254 (1953). (Russian)

For hypercomplex numbers of the form $\alpha = \alpha_0 + \sum_{k=1}^{m} \alpha_k \omega_k$, where $\alpha_0, \cdots, \alpha_m$ are ordinary complex numbers and the units ω_k satisfy $\omega_i \omega_j = 0$ (even for $i = j$), the author defines monogeneity in the usual way in terms of the limit of a difference quotient, obtains a necessary and sufficient condition for monogeneity, and shows that if $f(\beta)$ is a monogenic hypercomplex function of a hypercomplex variable, then all derivatives of $f(\beta)$ exist and are monogenic; explicit expressions are given. It is pointed out that many results in ordinary complex variable theory have analogues in the system being considered; in particular, an analogue of the familiar result concerning the vanishing of the Laplacian is given. *E. F. Beckenbach* (Los Angeles, Calif.).

Theory of Series

Gaier, Dieter. Zur Frage der Indexverschiebung beim Borel-Verfahren. Math. Z. **58**, 453–455 (1953).

The question is that of the equality of the Borel sums of the series $\sum_0^\infty a_n$ and $\sum_0^\infty a_{n-1}$ $(a_{-1} = 0)$ [Hardy, Divergent series, Oxford, 1949, pp. 182–195; these Rev. **11**, 25; Karamata, Math. Z. **45**, 635–641 (1939)]. The equality is established if $\sum_0^\infty a_n z^n$ has a positive radius of convergence. The argument depends on the lemma. If $f(z)$ is regular and of exponential type in $\Re |z| > 0$ and $f(x) \to 0$ as $x \to +\infty$ then $f'(x) \to 0$ as $x \to +\infty$. *A. J. Macintyre* (Aberdeen).

Wall, H. S. Hausdorff means with convex mass functions. Proc. Amer. Math. Soc. **4**, 637–639 (1953).

The author quotes the following criterion for a Hausdorff method $[H, \varphi]$ to include $(C, 1)$ summability:

$$(*) \qquad \varphi(x) = f(x) + x \int_x^1 \frac{1}{t} df(t),$$

where $[H, f]$ is regular [Garabedian, Hille and Wall, Duke Math. J. **8**, 193–213 (1941); these Rev. **2**, 278]. It may be remarked that the equation (*) is formally equivalent to:

$$(**) \quad \int_0^1 x^n d\varphi = \frac{1}{(n+1)} \int_0^1 x^n df \quad (\varphi(1) = f(1); n = 0, 1, \cdots);$$

$1/(n+1)$ being the moment sequence for $(C, 1)$ summability [compare Rogosinski, Proc. Cambridge Philos. Soc. **38**, 166–192 (1942); these Rev. **3**, 296].

Here the author shows that (*) is equivalent to the condition that $\varphi = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$, where φ_j $(j = 1, 2, 3, 4)$ is non-negative, non-decreasing and convex (upwards) on the interval $[0, 1]$. *L. S. Bosanquet* (London).

***Denjoy, Arnaud. Approximation sommatoire de certaines séries analytiques.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 89–94. Casa Editrice Perrella, Roma, 1953.

The purpose is the summation of a series $\sum u_n$ where $u_n = u(n)$ is a given function which is differentiable any number of times. The following formula is employed (f^{-k} is the kth integral)

$$f(x) = \sum_{k=1}^{p} (-1)^{k-1} \binom{p}{k} h^{-k} D_k[f^{-k}(x), h] + (-1)^p h^{-p} A_p[f(x), h]$$

where

$$D_q[f(x), h] = \sum_{k=0}^{q} (-1)^k \binom{q}{k} f(x + (q-k)h),$$

$$A_p[f(x), h] = \int_0^h \cdots \int_0^h (h - t_1) \cdots (h - t_p) f^p \times (x + t_1 + \cdots + t_p) dt_1 \cdots dt_p.$$

The resulting formulas are applied to $\sum n^{-s} = \zeta(s)$.

G. Szegö (Stanford, Calif.).

Levin, V. I. A limit estimate of the accuracy of asymptotic expansions of a certain class of functions. Trudy Moskov. Mat. Obšč. **2**, 383–395 (1953). (Russian)

For those functions $\varphi_t(t)$ which possess a Laplace transform $\varphi_t^*(p)$ of the following type

$$\varphi_t^*(p) = (p - p_0)^{\gamma - 1} \sum_1^\infty \frac{A_k}{\{(p - p_0)^\tau - \zeta^\tau\}^k},$$

where τ is positive rational and $> \gamma$, an asymptotic expansion can be derived. For the quotient of the remainder and the first omitted term estimates are given, using a finite part of the asymptotic series. *H. A. Lauwerier.*

Speigel, M. R. The summation of series involving roots of transcendental equations and related applications. J. Appl. Phys. **24**, 1103–1106 (1953).

Examples are given of the known relations between the sums of powers of reciprocals of roots of an equation $f(z) = 0$, $f(z)$ entire, and the coefficients of the Maclaurin expansion of $f(z)$. *P. W. Ketchum* (Urbana, Ill.).

Fourier Series and Generalizations, Integral Transforms

Izumi, S., Matsuyama, N., and Tsuchikura, T. Notes on Fourier analysis. XLIX. Some negative examples. Tôhoku Math. J. (2) **5**, 43–51 (1953).

If $f(t)$ is integrable and $\varphi(t) = f(x+t) + f(x-t) - 2f(x)$, then the statement $\int_0^t \varphi(u) du = o(1)$ is shown to be the best possible even at a point x at which the Fourier series of $f(t)$ converges. The theorem of Izumi [same J. (2) **1**, 285–302 (1950); these Rev. **12**, 405] which asserts that $\sigma_n(x) - f(x) = o(n^{\gamma - \beta})$ with $\beta > \gamma > -1$ implies $\Phi_n(t) = o(t^{n+\beta-\gamma})$

as $t\to 0$ for $\alpha>1+\gamma$, where $\Phi_\alpha(t)$ is the αth integral of $\varphi(u)$, is also shown to be best possible. *P. Civin.*

Bari, N. K. Generalization of inequalities of S. N. Bernšteĭn and A. A. Markov. Doklady Akad. Nauk SSSR (N.S.) **90**, 701–702 (1953). (Russian)

The following results are stated without proof. 1) Let $1\leqq p\leqq\infty$, $T_n(x)$ a trigonometric polynomial of order n, (a, b) a proper subinterval of $(0, 2\pi)$. Then, if $\|T_n\|_p\leqq M$ on (a, b), for every subinterval (a', b') of (a, b) we have $\|T_n\|_p\leqq C(a', b')nM$, where $C(a', b')$ is a constant depending on a', b' only. (For $p=\infty$ this result had been established by Privalov [Cauchy's integral, Saratov, 1919].) 2) If $\|T_n\|_p\leqq M$ on (a, b), then $\|T_n'\|\leqq Cn^2M$, where C depends on a and b only. (For $p=\infty$ the result is due to D. Jackson [Bull. Amer. Math. Soc. **37**, 883–890 (1931)].) 3) If $\|T_n(x)\sin x\|_p\leqq M$, then $\|T_n\|\leqq CnM$, where C is an absolute constant and the norms are taken on $(0, 2\pi)$ (for $p=\infty$, the result had been proved by S. Bernstein [Extremal properties of polynomials, v. 1, Leningrad-Moscow, 1937 (in Russian)]). *A. Zygmund* (Cambridge, England).

Mann, Paul August. Summation von Fourierschen Reihen mittels der Laplaceschen Transformation. Arch. Elektr. Übertragung **7**, 390–392 (1953).

Let $(1/2\pi)\sum_{-\infty}^{\infty}b(in)e^{inx}$ denote a Fourier series converging to a function $F(x)$ for all real x. If the function $b(z)$ is analytic in the finite plane of the complex variable z except for a finite number of poles, then for $0<x<2\pi$ the function $F(x)$ can be written in closed form, at least in some cases, as the sum of the residues of the function $b(z)e^{zx}(1-e^{2\pi z})^{-1}$ at the poles of $b(z)$. The author indicates this result by manipulations with impulse functions and divergent series, a process that does not produce conditions of validity. He presents two examples. *R. V. Churchill.*

Natanson, I. P. On convergence of certain interpolation processes for functions of two arguments. Mat. Sbornik N.S. **33(75)**, 219–232 (1953). (Russian)

Let $x_i^{(m)}=2i\pi/(2m+1)$ for $m=0, 1, 2, \cdots$ and $i=0, \pm1, \pm2, \cdots$. For a given function $f(x, y)$ of period 2π in x and y integrable R over any finite square, let $T_{m,n}(x, y)$ be the trigonometric polynomial of order $\leqq m$ in x and order $\leqq n$ in y, coinciding with f at the points $(x_i^{(m)}, x_k^{(n)})$. Such a polynomial exists and is unique. The main result of the paper asserts that $T_{m,n}(x_0, y_0)\to f(x_0, y_0)$ as m and n tend to infinity, provided the following conditions are satisfied: a) there exists a function $\Phi(x, y)$ defined in the square $Q=[x_0-\pi\leqq x\leqq x_0+\pi; y_0-\pi\leqq y\leqq y_0+\pi]$ such that

$$|f(x, y)-f(x_0, y)-f(x, y_0)+f(x_0, y_0)|\leqq\Phi(x, y)|x-x_0||y-y_0|;$$

b) the function $\Phi(x, y)$ decreases as x moves away from x_0 and y away from y_0; c) the integral $\int_Q\Phi dxdy$ (possibly improper) is finite; d) there exist functions $\varphi_1(x)$ and $\varphi_2(y)$ defined respectively in the intervals $(x_0-\pi, x_0+\pi)$ and $y_0-\pi, y_0+\pi$ decreasing as x moves away from x_0 and y from y_0, integrable over the intervals of definition and such that

$$|f(x, y_0)-f(x_0, y_0)|\leqq\varphi_1(x)|x-x_0|,$$
$$|f(x_0, y)-f(x_0, y_0)|\leqq\varphi_2(y)|y-y_0|.$$

This result is an analogue of the very well known extension to double Fourier series of the Dini test. The more complicated form of the extension is apparently essential; at least it is so in the one-dimensional case as proved previously by the author [see C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 53–56 (1944); these Rev. **6**, 125]. *A. Zygmund.*

Kovan'ko, A. S. On compactness of systems of generalized almost periodic functions of Weyl. Ukrain. Mat. Žurnal **5**, 185–195 (1953). (Russian)

The S_w-distance between two functions $f(x)$ and $\varphi(x)$ is defined by

$$D^TS_w(f, \varphi)=\sup_{-\infty<a<\infty}\left(\frac{1}{T}\int_a^{a+T}|f-\varphi|^w dx\right)^{1/w}.$$

When t tends to ∞, the S_w-distance tends to the W_w-distance. Let E be a subset of the real axis. If the integral in the S_w-distance is extended only over points of E in the interval $(a, a+T)$, we obtain another distance, which we denote $D^{TE}_{S_w}(f, \varphi)$. If $f-\varphi\equiv1$, this distance is the average density $\delta_S{}^TE$ of the set E. The sequence $f_1(x), f_2(x), \cdots$ is called D_{W_w}-uniformly convergent to $f(x)$ if (for $\epsilon>0$) there exists a number $T_0>0$ such that $\limsup_{n\to\infty}D^TS_w(f, f_n)<\epsilon$ when $T\geqq T_0$. The author proves that a set $\mathfrak{M}$ of W_w-almost periodic functions is compact with respect to D_{W_w}-uniform convergence if the following conditions are satisfied. 1) To $\epsilon>0$ correspond $\sigma>0$ and $T_1>0$ such that $D^{TE}_{S_w}(f, 0)<\epsilon$ if $\delta_S{}^TE<\sigma$, $T\geqq T_1$, and $f\in\mathfrak{M}$. 2) To $\epsilon>0$ correspond $\eta>0$ and T_2 such that $D^TS_w(f(x+h), f(x))<\epsilon$ if $|h|<\eta$, $T\geqq T_2$, and $f\in\mathfrak{M}$. 3) To $\epsilon>0$ correspond $T_3>0$ and a relatively dense set γ of real numbers such that $D^TS_w(f(x+\tau), f(x))<\epsilon$ if $\tau\in\gamma$, $T\geqq T_3$, and $f\in\mathfrak{M}$. The author applies this result to give a new proof of the approximation theorem for W_w-almost periodic functions. *H. Tornehave* (Copenhagen).

Dyson, F. J. Fourier transforms of distribution functions. Canadian J. Math. **5**, 554–558 (1953).

Let ϕ_n and ϕ be distribution functions, Φ_n and Φ their Fourier-Stieltjes transforms. It is well known that the uniform convergence of Φ_n to Φ implies pointwise convergence of ϕ_n to ϕ. The author proves that it also implies that the convergence of ϕ_n to ϕ is uniform with respect to x, but not necessarily uniform with respect to the whole class of ϕ's. In other words, given $\delta>0$, there exists a positive ϵ such that $|\Phi_n(t)-\Phi(t)|<\epsilon$ for all t implies $|\phi_n(x)-\phi(x)|<\delta$ for all x if ϵ is allowed to depend on δ and ϕ, but not if ϵ is required to depend on δ only. *R. P. Boas, Jr.*

Pollard, Harry. The harmonic analysis of bounded functions. Duke Math. J. **20**, 499–512 (1953).

The following is a famous unanswered question in harmonic analysis: Is every closed ideal in $L_1=L_1(-\infty, \infty)$ the intersection of the maximal ideals containing it? An equivalent question is the following: Is every weakly closed invariant subspace of $L_\infty=L_\infty(-\infty, \infty)$ spanned by the characters $e^{\Omega x}$ which it contains? Let $\varphi\in L_\infty$, let S_φ denote the smallest weakly closed invariant subspace of L_∞ containing φ, and let $\sigma(\varphi)$, the spectrum of φ, be the set of all λ such that $e^{\Omega x}\in S_\varphi$. [Reviewer's note: All known weakly closed invariant subspaces of L_∞ have the form S_φ.] Let $S_\varphi^\perp$ be the (closed ideal) of all $h\in L_1$ such that $h^*\varphi=0$. It can be shown that S_φ is spanned by the characters it contains if and only if $\hat{h}(\lambda)=\int_{-\infty}^{\infty}h(x)e^{\Omega x}dx=0$ for all $\lambda\in\sigma(\varphi)$ implies $h\in S_\varphi^\perp$. The paper under review is concerned with this implication for various classes of $\varphi\in L_\infty$ and $h\in L_1$. If $\hat{h}(\lambda)=0$ for $\lambda\in\sigma(\varphi)$ implies $h\in S_\varphi^\perp$ for all h in a subset R of L_1, then φ is said to admit R-synthesis. If φ is Bohr-almost-periodic or is a Fourier-Stieltjes transform, or if $\sigma(\varphi)$ has a countable boundary, then φ is known to admit L_1-synthesis [Eberlein, Proc. Symposium on Spectral Theory and Differential Problems, Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951, pp. 209–

219; these Rev. 13, 435; H. Helson, Ark. Mat. 1, 497–502 (1952); these Rev. 14, 246].

Simple proofs of all of these facts are given here, and two short elementary proofs of Wiener's Tauberian theorem are given in passing. The main new result is that every $\varphi \in L_\infty$ admits synthesis for the class of $h \in L_1$ whose Fourier transforms have modulus of continuity $\omega(\delta)$ such that $\omega = O(\delta^{1/2})$. The basic analytic device used is Bochner's 2-transform F for $\varphi \in L_\infty$:

$$F(t) = \int_{|x|\geq 1} \frac{e^{ixt}}{-x^2}\varphi(x)dx + \int_{-1}^{1} \frac{e^{ixt}-1-ixt}{-x^2}\varphi(x)dx.$$

The set $\sigma_1(\varphi)$ is defined as the complement of the set of all λ in some neighborhood of which $F(t)$ is linear, and a fundamental theorem shows that $\sigma(\varphi) = \sigma_1(\varphi)$. This equality is used very ingeniously in carrying out the various proofs. It is also shown that every closed subset of the line is $\sigma(\varphi)$ for some $\varphi \in L_\infty$ (this φ can be taken as an almost periodic function with absolutely convergent Fourier series).

A number of more or less obvious misprints were observed as well as a mistake in the proof of Lemma 4.1; the lemma is however true, and the needed correction to the author's proof is obvious. On lines 4 and 6 of page 505, it appears that "$\lambda \overset{*}{M} \varphi$" should be replaced by "$b \overset{*}{M} \varphi$". The reader is also cautioned against the misprint on line 4 of page 506.

E. Hewitt (Seattle, Wash.).

Takeda, Ziro. Note on Fourier-Stieltjes integral. II. Kōdai Math. Sem. Rep. **1953**, 33–36 (1953).

In a previous article [same Rep. **1952**, 59–61; these Rev. **14**, 371] the author considered the following theorem: Let $f(x)$ be a measurable function defined on a locally compact abelian group G and such that

$$|\sum c_\mu f(x_\mu)| \leq M \sup [|\sum c_\mu (x_\mu, \hat{x})| ; \hat{x} \in \hat{G}]$$

for each finite set of elements $x_\mu \in G$ and complex numbers c_μ. Then there exists a Radon measure $\hat{m}$ on $\hat{G}$ such that $f(x) = \int_{\hat{G}} (x, \hat{x}) d\hat{m}(\hat{x})$ a.e. On the real line the theorem was proved by S. Bochner for $f(x)$ continuous and by the reviewer for $f(x)$ measurable. The argument given in the above cited paper held only for $f(x)$ continuous. In the present article, the author gives a corrected version of the proof which takes care of measurable functions as well. The paper concludes with a generalization of a theorem due to H. Cramér, namely: Let $\phi_w(x)$, $w \in W$, be a directed set of continuous, positive definite, integrable functions which converge to 1 uniformly on each compact subset of G with $\phi_w(0) = 1$ for all $w \in W$. Then a measurable function $f(x)$ can be represented as a Fourier-Stieltjes integral a.e. if and only if $\psi_w(x) \equiv \int_G \overline{(x, \hat{x})} \phi_w(x) f(x) dx$ can be defined and $\int_{\hat{G}} |\psi_w(\hat{x})| d\hat{x} \leq M$ for each $w \in W$, where M is fixed.

R. S. Phillips (New Haven, Conn.).

Bhatnagar, K. P. On self-reciprocal functions. Ganita **4**, 19–37 (1953).

A function is said to be R_μ if it is self-reciprocal with respect to the Hankel kernel $(xy)^{1/2} J_\mu(xy)$; it is said to be $R_{\mu\nu}$ if it is self-reciprocal with respect to the kernel

$$(xy)^{1/2} \int_0^\infty J_\nu(xt) J_\mu(y/t) t^{-1} dt.$$

f and ϕ are connected by the relation $\phi(p) = \int_0^\infty g(pt) f(t) dt$. The author proves (more or less formally) a number of results of which we give a few samples (omitting conditions of validity). (1) Any two of the three properties (i) $f(t)$ is R_μ, (ii) $g(t) = t^{-1} g(t^{-1})$, (iii) $t^{-1}\phi(t^{-1})$ is R_μ implies the third. (2) Same result with f and ϕ interchanged. (3) If $t^{-1} f(t^{-1})$ is R_μ and $g(t)$ is R_ν then $\phi(t)$ is $R_{\mu\nu}$. (4) If $f(t)$ and $g(t)$ are R_μ then $\phi(p) = p^{-1}\phi(p^{-1})$. The results are illustrated by examples.

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Sharma, Ambikeshwar. The complex zeros of a polynomial. Math. Student **21**, 52–54 (1953).

The author proves that the zeros of

$$f(z) = \sum_{k=0}^{n} [1/(m+k)!](k+a+m)^r z^k$$

lie in the region $b_m \leq |z| \leq b_{m+n-1}$ where

$$b_m = (m+1)(a+m)^r (a+m+1)^{-r}$$

by showing that $|(b_m - z) f(z)| > 0$ for $|z| < b_m$ and

$$|(1-z) z^n f(b_{m+n-1}/z)| > 0 \quad \text{for} \quad |z| < 1.$$

As an application, he sets $x^{-r}(e^x - 1)^r = \sum_{n=0}^{\infty} P_n(r) x^n$; then the zeros of $f(z) = \sum_{k=0}^{n} P_{m+k}(r) z^k$ lie in

$$P_m(r)/P_{m+1}(r) \leq |z| \leq P_{m+n-1}(r)/P_{m+n}(r).$$

M. Marden (Milwaukee, Wis.).

Novoselov, V. S. Necessary and sufficient conditions that the roots of a polynomial not have positive real parts and that the multiplicity of the zero and imaginary roots not exceed a given number. Mat. Sbornik N.S. **33(75)**, 215–218 (1953). (Russian)

The classic necessary and sufficient conditions of Hurwitz that a polynomial have all roots in the open left half-plane, are now sharpened to include the case where some of the roots may lie on the imaginary axis.

A. W. Goodman.

Faedo, Sandro. Un nuovo problema di stabilità per le equazioni algebriche a coefficienti reali. Ann. Scuola Norm. Super. Pisa (3) **7**, 53–63 (1953).

Let $f(z)$ be a polynomial $a_0 + a_1 z + \cdots + a_n z^n$ with real coefficients. If it is possible to find real numbers $\underline{a}_i$, $\bar{a}_i$, $i = 0, \cdots, n$, which are certain functions of the coefficients a_i, such that $\underline{a}_i \leq a_i \leq \bar{a}_i$, this condition is sufficient for $f(z)$ to be a Hurwitz polynomial. If $\underline{a}_i = \bar{a}_i$, the above condition is equivalent to the Routh criterion.

E. Frank.

Wenzl, F. Nullstellendichte reeller Polynome und Tschebyscheffsche Approximation. Math. Z. **59**, 17–39 (1953).

Let $g(x)$ be a positive continuous function in $[-1, +1]$. The author is interested in the following problem of the Tchebychev type:

$$\min |f(x)| = \min \{g(x) |x^n + a_1 x^{n-1} + \cdots + a_n|\},$$

$-1 \leq x \leq 1$. The solution is characterized (as known) by the fact that $f(x)$ assumes its maximum modulus $n+1$ times with alternate signs. Let ξ_ν be the roots of $F(x)$. The author introduces the function $\rho_n(\xi)$ defined by the condition $n\rho_n(\xi) = (\xi_{\nu+1} - \xi_\nu)^{-1}$ for $\xi_\nu \leq \xi < \xi_{\nu+1}$. Assuming that $\rho_n(\xi) \to \rho(\xi)$ uniformly in every interval $|\xi| \leq 1 - \epsilon$, $\epsilon > 0$, and $|\rho_n(\xi)| < M(1-\xi^2)^\alpha$, $\alpha > -1$, it is proved that the expression

$$\int_{-1}^{+1} \log|x - \xi| \rho(\xi) d\xi$$

is constant for $-1<x<1$. This result suggests a method for the approximate solution of the Tchebychev problem which is illustrated in the special case $g(x)=e^{-x}$, $n=6$. This method is compared with that of Remes [C. R. Acad. Sci. Paris **199**, 337–340 (1934)]. *G. Szegö.*

***Lorentz, G. G. Bernstein polynomials.** Mathematical Expositions, no. 8. University of Toronto Press, Toronto, 1953. x+130 pp. $5.75.

This monograph deals with the study of remarkable polynomials

$$B_n(x)=\sum_{v=0}^{n} f\left(\frac{v}{n}\right)\binom{n}{v}x^v(1-x)^{n-v}$$

associated with a given function $f(x)$, $0\leqq x\leqq 1$. Suggested by probability considerations they were introduced by S. Bernstein in 1912–13; with their aid Bernstein gave a particularly elegant proof of Weierstrass' theorem which is the starting point of the present book. The degree of approximation is studied and also the mode of approximation for various classes of $f(x)$. In Chapter 2 generalizations of the Bernstein polynomials and their relation to methods of summability are considered. Chapter 3 deals with function spaces and moment problems. In Chapter 4 the convergence of Bernstein polynomials for analytic functions in the complex domain is studied in detail. The fundamental convergence theorem (p. 104) is an important result of this chapter. A rather exhaustive bibliography closes the book. The subject is an interesting and suitable testing ground for many concepts and methods of modern analysis.

G. Szegö (Stanford, Calif.).

Kakehashi, Tetsujiro. Interpolation on the real axis. J. Osaka Inst. Sci. Tech. Part I. **3**, 3–19 (1951).

Following the investigations of Walsh [Interpolation and approximation . . . , Amer. Math. Soc. Colloq. Publ., v. 20, New York, 1935], the author studies the interpolation polynomials

$$S_n(z:f)=\sum_{k=0}^{n} a_k(z-x_1)(z-x_2)\cdots(z-x_k),$$

$$a_k=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-x_1)(t-x_2)\cdots(t-x_{k+1})},$$

where $f(z)$ is a given function analytic within a curve C containing $[-1,+1]$ in its interior. Assuming the uniform convergence of the sequence of distribution functions $\mu_n(x)$ having equal jumps at x_k, $k=1, 2, \cdots, n$, i.e. $\lim\mu_n(x)=\mu(x)$, the function

$$\phi(z)=\exp\left\{\int_{-1}^{+1}\log|z-x|\,d\mu(x)\right\}$$

determines a set of "lemniscates" $\phi(z)=R$. The domain of convergence of $S_n(z;f)$ is a lemniscate determined similarly as in the case of power series. Generalizations are considered when the points of interpolation $x_k=x_{kn}$ depend on n. Finally interpolations in the zeros of certain orthogonal polynomials are studied. *G. Szegö* (Stanford, Calif.).

Šaginyan, A. L. On best approximations by harmonic polynomials in space. Doklady Akad. Nauk SSSR (N.S.) **90**, 141–144 (1953). (Russian)

The author studies the relationship between the best approximation E_n of continuous functions on a closed smooth surface S by means of harmonic polynomials $H_n(x, yz)$ of degree n and the modulus of continuity of the function. The following theorem illustrates the type of result obtained. Let the curvatures $1/R_1$, $1/R_2$ of S satisfy $1/R_1^2+1/R_2^2\leqq$ const. If f is an arbitrary continuous function on S for which $E_n(f)<w(n)/n^p$ where $w(x)$ is an ultimately non-increasing function, $w(x)\to 0$ as $x\to\infty$ and $\int^{\infty}w(x)x^{-1}dx<\infty$, then f is the boundary value of a function which is continuous on the closed region and harmonic in the interior and which possesses a derivative $\partial^p F/\partial l_1\partial l_2\cdots\partial l_p$ with a modulus of continuity

$$w(\delta)\leqq a\left[\delta\int_a^{a/\delta}w(x)dx+\int_{1/\delta}^{\infty}w(x)x^{-1}dx\right].$$

$l_1, l_2, \cdots$ are arbitrary directions and a and a are constants which are independent of δ. *P. Davis.*

Vekua, I. N. On completeness of a system of harmonic polynomials in space. Doklady Akad. Nauk SSSR (N.S.) **90**, 495–498 (1953). (Russian)

The harmonic polynomials

$$U_{nm}=r^nP_{nm}(\cos\theta)\begin{matrix}\cos\phi\\ \sin\phi\end{matrix}\quad(m\leqq n)$$

are proved complete on smooth surfaces S. The author then orthogonalizes this system on S and proves the usual type of expansion and approximation theorem for harmonic functions. *P. Davis* (Washington, D. C.).

Vekua, I. N. On completeness of a system of metaharmonic functions. Doklady Akad. Nauk SSSR (N.S.) **90**, 715–718 (1953). (Russian)

The same treatment is given to the system

$$U_{nm}=(\lambda r)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\lambda\gamma)P_{nm}(\cos\theta)\begin{matrix}\cos m\phi\\ \sin m\phi\end{matrix}\quad(m\leqq n)$$

which are solutions of $\Delta u+\lambda^2u=0$ (cf. the preceding review). *P. Davis* (Washington, D. C.).

Izumi, Shin-ichi. On an approximation problem in the theory of probability. Tôhoku Math. J. (2) **5**, 22–28 (1953).

Let (a) $\sum_{k=0}^{\infty}|f(k)|w^{-k}$ converge for some w, or (a′) $\sum_{k=0}^{\infty}(f(k))^2w^{-k}$ converge for some positive w. Then, given any $\epsilon>0$, (b) there is a polynomial $P(x)$ such that

$$\sum_{n=0}^{\infty}|P(n)-f(n)|\lambda^n/n!<\epsilon,$$

or (b′) $\sum_{n=0}^{\infty}(P(n)-f(n))^2\lambda^n/n!<\epsilon$. The proof makes use of the Szász form of the Bernstein polynomial for $f(k)$.

P. Davis (Washington, D. C.).

Geronimus, Ya. L. On asymptotic properties of polynomials which are orthogonal on the unit circle, and on certain properties of positive harmonic functions. Amer. Math. Soc. Translation no. **95**, 29 pp. (1953).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 123–144 (1950); these Rev. **12**, 177.

Tortrat, A. Les fonctions orthogonales d'Hermite et les relations d'incertitude. J. Math. Pures Appl. (9) **32**, 85–128 (1953).

The first two chapters of this memoir are expanded versions of two notes [C. R. Acad. Sci. Paris **226**, 298–300, 543–545, 758–759 (1948); these Rev. **9**, 349], and the third

chapter shows the application of Hermite polynomials to the discussion of Heisenberg's uncertainty relation.

A. Erdélyi (Pasadena, Calif.).

Dougall, John. The product of two Legendre polynomials. Proc. Glasgow Math. Assoc. **1**, 121–125 (1953).

This paper contains a new derivation of the coefficients in the expansion into a series of Legendre polynomials of the product of two Legendre polynomials. The author's ingenious proof is based on the repeated application of Laplace's operator to homogeneous harmonic polynomials.

A. Erdélyi (Pasadena, Calif.).

Special Functions

Fogel' [Fogelis], È. K. A finite theory of elementary functions. I. Logarithmic and exponential functions. Latvijas PSR Zinātņu Akad. Vēstis **1951**, no. 5 (46), 801–813 (1951). (Russian. Latvian summary)

The author develops "finite" analogues of such functions as e^x, x^a, and $\log x$. This consists essentially of working, for x in a given interval, with suitable partial sums of the well known power series expansions of these functions, and examining to what extent these partial sums inherit approximations of certain well known properties of these functions. For example, considering x in the interval $[-X, X]$ the function $E(x)=1+x+x^2/2!+\cdots+x^M/M!$, where $M=4X^2$, serves as the "finite" analogue of e^x. Similar constructions are given for "finite" analogues of $\log x$ and x^a, and the relationships between them investigated.

H. N. Shapiro (New York, N. Y.).

Agarwal, R. P. On integral analogues of certain transformations of well-poised basic hypergeometric series. Quart. J. Math., Oxford Ser. (2) **4**, 161–167 (1953).

Integral analogues of certain transformations of generalized hypergeometric series are given in W. N. Bailey's "Generalized hypergeometric series" [Cambridge, 1935]. The author gives the corresponding integral analogues for transformations of basic hypergeometric series. The integrals involved are of the type introduced by G. N. Watson [Trans. Cambridge Philos. Soc. **21**, 281–299 (1910)], and they are further investigated in the present paper.

A. Erdélyi (Pasadena, Calif.).

Bailey, W. N. On the sum of a terminating ${}_3F_2(1)$. Quart. J. Math., Oxford Ser. (2) **4**, 237–240 (1953).

The author proves the formula

$$ {}_3F_2\left[\begin{matrix} a, b, -m \\ \frac{1}{2}(a+b+1), -2m \end{matrix}\right]=\frac{(\frac{1}{2}+\frac{1}{2}a)_m(\frac{1}{2}+\frac{1}{2}b)_m}{(\frac{1}{2})_m(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)_m} $$

and some other results on terminating ${}_3F_2$. He deduces formulas which express hypergeometric series of two variables for special values of the variables ($y=-x$) in terms of hypergeometric series of one variable.

A. Erdélyi.

Humbert, Pierre, et Delerue, Paul. Sur une extension à deux variables de la fonction de Mittag-Leffler. C. R. Acad. Sci. Paris **237**, 1059–1060 (1953).

The authors introduce the function

$$ E_{\alpha,\beta}(x, y)=\sum_m\sum_n \frac{x^{[m\alpha+(n+1)\beta-1]}y^n}{\Gamma[m\alpha+(n+1)\beta]\Gamma(n\beta+1)}, $$

Mittag-Leffler's function being $E_\alpha(x)=E_{\alpha,1}(x, 0)$. They obtain the operational representation of $E_{\alpha,\beta}(x^\alpha, y^\beta)$, derive a functional relation satisfied by $E_{\alpha,\beta}(x, y)$, and point out that $E_\alpha(x)+E_\alpha(-x)=2E_{2\alpha}(x^2)$, this being a generalization to any α of $e^x+e^{-x}=2\cosh x$.

A. Erdélyi.

MacRobert, T. M. Integral of an E-function expressed as a sum of two E-functions. Proc. Glasgow Math. Assoc. **1**, 118 (1953).

Evaluation of

$$ \int_0^1 t^{\rho}(1-t)^{\sigma-\alpha-1}E(p; \alpha_r: q; \rho_s: xt)dt $$

in terms of E-functions. [Reviewer's remark: see also Erdélyi et al., Higher transcendental functions, vol. I, McGraw-Hill, New York, 1953, p. 214 equation (5).]

A. Erdélyi (Pasadena, Calif.).

MacRobert, T. M. An integral involving an E-function and an associated Legendre function of the first kind. Proc. Glasgow Math. Assoc. **1**, 111–114 (1953).

Evaluation, in terms of E-functions, of the integral

$$ \int_{-1}^1 (1-\lambda^2)^{\frac{1}{2}m}T_{m+n}^{-m}(\lambda)(1-\lambda)^{-l-m-1} \times E(l+m+1, \alpha_1, \cdots, \alpha_p: \rho_1, \cdots, \rho_q: (1-\lambda)z)d\lambda, $$

where T denotes associated Legendre functions on the cut, and E is the author's generalization of hypergeometric functions.

A. Erdélyi (Pasadena, Calif.).

Ragab, Fouad M. An integral involving a modified Bessel function of the second kind and an E-function. Proc. Glasgow Math. Assoc. **1**, 119–120 (1953).

Evaluation of

$$ \int_0^\infty K_{mn}(2m\lambda)\lambda^{mk-1}E(p; \alpha_r: q; \beta_s: x\lambda^{-2m})d\lambda $$

for positive integer m. This result generalizes an earlier formula (for $m=2^r$) by the same author [same Proc. **1**, 72–75 (1952); these Rev. **14**, 468].

A. Erdélyi.

Ragab, Fouad M. Generalisation of an integral due to Hardy. Proc. Glasgow Math. Assoc. **1**, 115–117 (1953).

Ragab, Fouad M. Integrals involving E-functions. Proc. Glasgow Math. Assoc. **1**, 129–136 (1953).

In the first of these papers the author proves

$$ \prod_{s=1}^{p-1}\int_0^\infty K_n(t_s)t_s^{2s/p-1}dt_s K_n\left(\frac{b}{t_1t_2\cdots t_{p-1}}\right)=\pi^{p-1}K_{pn}(pb^{1/p}) $$

and a similar result for Bessel functions of the first kind, and in the second paper, the corresponding result with $E(\alpha, \beta::z)$ replacing $K_n(z)$ in the integrand.

A. Erdélyi.

Srivastava, H. M. On certain connections between the generalised K-function of Bateman and Legendre and parabolic cylinder functions. Bull. Calcutta Math. Soc. **44**, 59–62 (1952).

The functions introduced here are confluent hypergeometric functions [see for instance Erdélyi et al., Higher transcendental functions, vol. I, McGraw-Hill, New York, 1953, p. 274, equation (13)] and the properties investigated here are known properties of confluent hypergeometric functions.

A. Erdélyi (Pasadena, Calif.).

Newton, T. D. Coulomb functions for large values of the parameter η. Atomic Energy of Canada Limited, Division of Research, Chalk River, Ontario, Rep. **CRT-526**, 10 pp. (1952).

In order to obtain asymptotic formulas, for large η, of Coulomb wave functions near the transition point, the author evaluates the integrals representing these functions asymptotically, using a modification of the saddle-point method. The results thus obtained are compared with the numerical results of Barfield and Broyles [Physical Rev. (2) **88**, 892 (1952); these Rev. **14**, 500], the agreement being excellent. *A. Erdélyi* (Pasadena, Calif.).

Wannier, Gregory H. Connection formulas between the solutions of Mathieu's equation. Quart. Appl. Math. **11**, 33–59 (1953).

Let

$$\text{(1)} \qquad \mathrm{je}^{+}(x) \quad \text{and} \quad \mathrm{je}^{-}(x)=\mathrm{je}^{+}(-x)$$

be the Floquet solutions of Mathieu's differential equation

$$\text{(2)} \qquad \frac{d^2t}{dx^2}+(a-2q\cos 2x)f=0,$$

i.e., solutions of (2) which obey the relation

$$\mathrm{je}^{+}(x+\pi)=e^{i\pi\beta}\mathrm{je}^{+}(x), \quad \mathrm{je}^{-}(x+\pi)=e^{-i\pi\beta}\mathrm{je}^{-}(x).$$

Furthermore, replace in (2) x by $\frac{1}{2}\pi+iy$ and denote a solution of

$$\text{(3)} \qquad \frac{d^2t}{dy^2}-(a+2q\cosh 2y)f=0$$

which behaves like $q^{-1/4}e^{-\nu}\exp(-q^{1/2}e^{y})$ for $y\gg 0$ by

$$\text{(4)} \qquad \mathrm{ke}\, y.$$

Then the four functions defined by

$$\text{(5)} \qquad \begin{aligned} \mathrm{he}^{(1)}\left(\frac{\pi}{2}+iy\right)&=\mathrm{ke}\, y\\ \mathrm{he}^{(2)}\left(-\frac{\pi}{2}+iy\right)&=\mathrm{ke}\, y\\ \mathrm{he}^{(3)}\left(-\frac{\pi}{2}-iy\right)&=\mathrm{ke}\, y\\ \mathrm{he}^{(4)}\left(\frac{\pi}{2}-iy\right)&=\mathrm{ke}\, y \end{aligned}$$

are solutions of (2) having given asymptotic character. Connection formulas between solutions of the type (1) and (5) are given, such as

$$\mathrm{je}^{\pm}(x)=\tfrac{1}{2}i\exp(\mp\tfrac{1}{2}i\pi\gamma)[\exp(\mp\tfrac{1}{2}i\pi\beta)\,\mathrm{he}^{(1)}(x) -\exp(\pm\tfrac{1}{2}i\pi\beta)\,\mathrm{he}^{(2)}(x)],$$

$$\mathrm{je}^{\pm}(x)=\tfrac{1}{2}i\exp(\pm\tfrac{1}{2}i\pi\gamma)[\exp(\pm\tfrac{1}{2}i\pi\beta)\,\mathrm{he}^{(3)}(x) -\exp(\mp\tfrac{1}{2}i\pi\beta)\,\mathrm{he}^{(4)}(x)],$$

$$\mathrm{he}^{(1)}(x)=\operatorname{cosec}(\pi\beta)\{\mathrm{je}^{+}(x)\exp[\tfrac{1}{2}i\pi(\gamma-\beta)] -\mathrm{je}^{-}(x)\exp[-\tfrac{1}{2}i\pi(\gamma-\beta)]\},$$

$$\mathrm{he}^{(1)}(x)=ie^{\Phi}\,\mathrm{he}^{(3)}(x)-[ie^{\Phi}\cos(\pi\beta)+\cos(\pi\gamma)]\,\mathrm{he}^{(4)}(x),$$

and the relation between β, γ, and Φ, all of which are functions of the parameters a and q of (2), is $i\sin(\pi\gamma)=e^{\Phi}\sin(\pi\beta)$. This parameter Φ is shown to be a slowly varying function of a and q, and maps of the upper and of the lower bound of Φ are given. *F. Oberhettinger* (Washington, D. C.).

Abramowitz, Milton. Evaluation of the integral

$$\int_0^\infty e^{-u^2-x^2/u}du.$$

J. Math. Physics **32**, 188–192 (1953).

For the integral mentioned in the title, the author obtains an expansion in ascending powers of x, and an asymptotic expansion in descending powers of x. An expansion of the latter type is also given for

$$f_m(x)=\int_0^\infty u^m\exp(-u^2-x/u)du.$$

Some misprints and errors in previous work concerning these functions are noted and corrected. *A. Erdélyi.*

Moon, Parry, and Spencer, Domina Eberle. Some coordinate systems associated with elliptic functions. J. Franklin Inst. **255**, 531–543 (1953).

Harmonic Functions, Potential Theory

Dinghas, Alexandre. Sur la croissance de certaines classes de fonctions sousharmoniques bornées sur des multiplicités données. C. R. Acad. Sci. Paris **237**, 690–691 (1953).

Estimates of the growth at infinity of functions u satisfying $\Delta u\geqq cu$ for given $c\geqq 0$. *P. R. Garabedian.*

Hausner, Melvin. Dirichlet's principle and generalized boundary values. Ann. of Math. (2) **57**, 475–489 (1953).

Soit D une variété différentiable C^∞ de dimension n, paracompacte et orientable. Soient Ω^r l'espace des formes différentielles C^∞ de degré r, à support compact, et Γ^{n-r} l'espace des "courants" de degré $n-r$ [un tel courant est une forme linéaire sur Ω^r, continue pour une topologie convenable; cf. De Rham et Kodaira, Harmonic integrals, Inst. for Advanced Study, Princeton, 1950; ces Rev. **12**, 279]. Dans $\Omega=\sum_r\Omega^r$, soit LGZ le noyau de d, et GZ l'image de d; dans $\Gamma=\sum_r\Gamma^r$, soit LG le noyau de d, et G l'image de d. On sait [loc. cit., p. 40, theorem C] que le sous-espace orthogonal à GZ^r est LG^{n-r}, et le sous-espace orthogonal à LGZ^r est G^{n-r}.

Supposons maintenant D munie d'une métrique riemannienne. Soit δ l'opérateur transposé de d; dans Ω, soit LCZ le noyau de δ, CZ l'image de δ; dans Γ, soit LC le noyau de δ, C l'image de δ. La métrique définit un isomorphisme $*$ de Ω^r sur Ω^{n-r}; pour l'application $(\omega, T)\to T[{}^*\omega]$ de $\Omega^r\times\Gamma^r$ dans les scalaires, le sous-espace orthogonal à GZ^r (resp. CZ^r, LGZ^r, LCZ^r) est LC^r (resp. LG^r, C^r, G^r).

Dans le présent article, l'auteur ne considère pas de courants en général; mais, pour chaque couple de nombres réels p, q tels que $1/p+1/q=1$, il considère l'adhérence, dans l'espace L^p des formes différentielles sur D, des sous-espaces GZ, CZ, LGZ, LCZ, et l'intersection, avec L^q, des espaces LC, LG, C, G. Ces 8 sous-espaces sont deux à deux orthogonaux, dans la dualité canonique entre L^p et L^q: l'auteur le prouve à nouveau, sans parler de courants, mais en référant au th. C de De Rham. L'auteur utilise un procédé de régularisation qui l'oblige à supposer que D possède un système de coordonnées locales $x_1, \cdots, x_n$ partout valable et pour lequel le ds^2 est égal à $\sum_i(dx_i)^2$. En fait, De Rham a prouvé, pour toute variété différentiable, l'existence de régularisations qui commutent avec d (résultat inédit); on

peut d'ailleurs se passer des régularisations pour démontrer les théorèmes d'orthogonalité ci-dessus.

En prenant $p=q=2$, l'auteur obtient facilement des décompositions orthogonales dans l'espace L^2. Puis il introduit l'espace vectoriel, quotient de $LG\cap L^2$ par l'adhérence (dans L^2) de LGZ; les éléments de cet espace quotient sont appelés les "composantes tangentielles à la frontière". De même, l'espace quotient de $LC\cap L^2$ par l'adhérence de LCZ est appelé l'espace des "composantes normales à la frontière". Les relations d'orthogonalité conduisent aussitôt au résultat suivant (cf. th. 14) : l'espace des formes harmoniques de L^2 (formes $\omega \in L^2$ telles que $d\omega=0$ et $\delta\omega=0$) est somme directe de deux sous-espaces orthogonaux, dont l'un est isomorphe à l'espace de L^2-cohomologie de la variété D, et l'autre à l'espace des composantes normales à la frontière (j'appelle espace de L^2-cohomologie l'espace des classes de cohomologie des éléments $\omega \in L^2$ tels que $d\omega=0$). Enoncé analogue avec la L^2-homologie et les composantes tangentielles à la frontière.

Si la variété D est connexe et "prolongeable", la métrique étant supposée analytique, alors tout courant adhérent à LCZ et à LGZ est nul; l'auteur le prouve pour les courants qui appartiennent en outre à L^p (th. 17); même résultat si D est un ouvert de l'espace euclidien E^n (th. 18), parce que toute forme harmonique de E^n, qui appartient à L^p, est nulle.

Erratum: p. 489, ligne 9, au lieu de "theorem 9", lire "theorem 10".

H. Cartan (Paris).

*Shiffman, Max. **On Dirichlet's principle.** Contributions to the theory of Riemann surfaces, pp. 49–53. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

Soit B le domaine extérieur à une courbe simple close B^* avec dérivées troisièmes continues, $F(p, q)$ une fonction convexe satisfaisant à

$$2K(\alpha^2+\beta^2) \geqq F_{pp}(p, q)\alpha^2+2F_{pq}(p, q)\alpha\beta + F_{qq}(p, q)\beta^2 \geqq 2k(\alpha^2+\beta^2)$$

(k, K positifs). Le problème qui consiste à rendre minimum

$$(1)\quad \iint_B [F(\psi_x, \psi_y) - F(p_0, q_0) - F_p(p_0, q_0)(\psi_x - p_0) - F_q(p_0, q_0)(\psi_y - q_0)]dxdy$$

pour les ψ s'annulant sur B^* a une solution unique avec dérivées secondes continues [Haar, Math. Ann. **97**, 124–158 (1926)]. L'auteur énonce le théorème suivant: Pour cette solution, ψ_x et ψ_y sont continues dans la fermeture de B et à l'infini et dépendent continuement de p_0, q_0.

Le cas où F est de la forme $h(\Xi)$ avec $\Xi=\psi_x^2+\psi_y^2$, et où $p_0=0$ correspond à l'écoulement d'un fluide compressible; d'où le résultat suivant: Il y a une constante critique $\bar{q}_0$ telle que, pour $0<q_0<\bar{q}_0$, il existe un écoulement de fluide compressible autour de B^*, ψ étant la fonction de courant, avec $\psi_x=0$, $\psi_y=q_0$ à l'infini, et le potentiel des vitesses étant uniforme. Pour cet écoulement, soit $V(q_0)$ le maximum de $\Xi^{1/2}$; quant q_0 varie de 0 à $\bar{q}_0$, $V(q_0)$ prend toutes les valeurs de 0 à b_0, b_0 étant la valeur de $\Xi^{1/2}$ pour laquelle l'équation en ψ cesse d'être complètement elliptique (écoulement subsonique). L'auteur étend ce dernier résultat au cas où la circulation n'est pas nulle. Il faut remplacer (1) par une intégrale plus compliquée.

R. de Possel (Alger).

*Rosenbloom, P. C. **Mass distributions and their potentials.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 130–138. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Inégalités variées entre les bornes inférieures d'une fonction surharmonique U sur certains ensembles, et la charge d'un autre ensemble (pour les masses associées à U); applications à des résultats connus et nouveaux sur la distribution des zéros d'une fonction analytique, notamment: soient $0<r_1<r_2<r_3<1$, $a>0$, $0<\tau\leqq 2\pi$; il existe trois constantes C, C_2, C_3, dépendant seulement de r_1, r_2, r_3, a et τ, telles que si $f(z)$ est analytique dans le cercle-unité, et si on a, en posant $M(r)=\mathrm{Max}_{|z|=r}|f(z)|$: $M(r_1)\geqq a$, $M(r_2)\leqq 1$, $M(r_3)\geqq M(1)^C$, $M(1)\geqq C_2$, alors le nombre des zéros de f dans tout angle de sommet l'origine et d'ouverture τ est au moins égal à $C_3 \log M(1)$. On trouvera une autre démonstration et des précisions sur ce "théorème des 4 cercles" dans un travail ultérieur de l'auteur [Bull. Soc. Math. France **79**, 1–58 (1951); **80**, 183–215 (1952); ces Rev. **15**, 233].

J. Deny (Strasbourg).

Koševlev, A. I. Differentiability of solutions of certain problems of potential theory. Mat. Sbornik N.S. **32(74)**, 653–664 (1953). (Russian)

S. G. Mihlin [Doklady Akad. Nauk SSSR (N.S.) **78**, 443–446 (1951); these Rev. **13**, 16] showed that if the function $f \in L^2(\Omega)$, where Ω is a plane domain with a sufficiently smooth boundary Γ, then the generalized solution of the Poisson equation $\partial^2u/\partial x_1^2+\partial^2u/\partial x_2^2=f$, in Ω, subject to the boundary condition $u=0$, on Γ, possesses generalized second derivatives which satisfy

$$\iint_\Omega \left|\frac{\partial^2 u}{\partial x_i \partial x_k}\right|^2 dx_1dx_2 \leqq C_1 \iint_\Omega |f|^2 dx_1dx_2, \quad i=1, 2,$$

where C_1 is a constant independent of f. In the first section of the present paper the author considers similar questions when $f \in L^p(\Omega)$, $p>1$, treating also the case of domains Ω which can be mapped conformally onto the unit disk by means of sufficiently smooth functions. In the second section the author proves several theorems of a similar nature concerning the dependence of the second partial derivatives of the solution of the Dirichlet problem on the differentiability properties of the prescribed boundary values and on the domain Ω. *J. B. Diaz* (College Park, Md.).

*Schröder, Kurt. **Untersuchungen über biharmonische Funktionen.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 571–584. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

The author surveys his previous results [Math. Z. **48**, 553–675 (1943); these Rev. **5**, 67] concerning existence and uniqueness of the solution of the fundamental boundary-value problems (both interior and exterior) for the biharmonic equation in the plane and in three-space. The method used is that of integral equations. These results enable the formulation of sufficient conditions for the existence of higher partial derivatives of a biharmonic function on the boundary of the domains considered. The last section is devoted to an explicit solution of a biharmonic boundary value problem for a rectangular domain [Schröder, Math. Ann. **121**, 247–326 (1949); these Rev. **12**, 185].

J. B. Diaz (College Park, Md.).

Pini, Bruno. Osservazioni su un teorema di M. Picone relativo all'equazione $\Delta u+cu=0$. Boll. Un. Mat. Ital. (3) 8, 19–25 (1953).

D'après Picone [Ann. Soc. Polon. Math. 21, 161–169 (1949); ces Rev. 11, 361] une solution de $\Delta u+c(M)u=0$ dans l'espace à $n\geqq 2$ dimensions qui s'annule à la frontière d'un domaine d'existence borné, sauf en quelques points P_i où elle ne croit pas trop vite (par exemple moins vite que $\overline{P_iM}^{-\alpha_i}$ $(\alpha_i>0)$ pour $n>2$), doit être nulle si $c(M)$ admet une borne supérieure convenable >0. Après Leja [ibid. 21, 170–172 (1949); ces Rev. 11, 361] qui apporta de légères améliorations, l'auteur remplace les conditions de limitation de croissance par des conditions en moyenne sur de petites sphères ou circonférences. Il remarque aussi que le raisonnement de Picone fournit des résultats analogues lorsque $c(M)$ admet certaines majorations qui n'incluent pas une borne supérieure finie. *M. Brelot* (Grenoble).

***Cheng, N. T. On a theorem of Nicolesco and generalized Laplace operators.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 771–773. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

The results contained in this note have already appeared in Proc. Amer. Math. Soc. 2, 77–86 (1951); these Rev. 12, 825. *M. O. Reade* (Ann Arbor, Mich.).

Satô, Yasuo. Transformation of spherical solid harmonics related to the translation of coordinates system. Bull. Earthquake Res. Inst. Tokyo 31, 203–210 (1953). (Japanese summary)

The author obtains transformation formulas for solid spherical harmonics when the origin of the spherical polar coordinates is shifted, by the amount a, on the polar axis. If R, θ, ϕ and R', θ', ϕ' are the two systems $(\phi=\phi')$ the transformation formula for internal harmonics is

$$R'^n P_n^m(\cos\theta') \begin{matrix}\cos\\ \sin\end{matrix} m\phi' = \sum_{k=m}^{n} \binom{n+m}{k+m} (-a)^{n-k} R^k P_k^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi.$$

There is a similar transformation for external harmonics

$$R^{-n-1} P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

except that there we have infinite series and the cases $R<a$ and $R>a$ must be distinguished. *A. Erdélyi.*

Differential Equations

Viktorovskiĭ, E. E. On a generalization of the concept of integral curves for a discontinuous field of directions. Doklady Akad. Nauk SSSR (N.S.) 89, 593–596 (1953). (Russian)

Let $f_i(x, y_1, \cdots, y_n)$ be measurable functions in an $(n+1)$-dimensional region G, such that

$$M(x)=\sup_i \operatorname{ess.sup}_{yi} |f_i(x, y_1, \cdots, y_n)|$$

is summable. A generalized solution of the system $y_i'=f_i$ through the initial point $(x_0, y_{10}, \cdots, y_{n0}) \in G$ is a system of n absolutely continuous functions $u_i(x)$ defined in X: $[x_0, x_0+\alpha]$, $u_i(x_0)=y_{i0}$, satisfying the following condition: given $\epsilon>0$ and any set N of $(n+1)$-measure 0, there are n^2 measurable functions $\psi_{ij}(x)$ such that in X:

$$f_i(x, \psi_{1i}, \cdots, \psi_{ni}) \text{ are summable;} \tag{1}$$

$$|u_i(x)-\psi_{ij}(x)|<\epsilon; \tag{2}$$

$$\left|u_i(x)-y_{i0}-\int_{x_0}^{x} f_i(x, \psi_{1i}, \cdots)dx\right|<\epsilon; \tag{3}$$

$$(x, \psi_{1i}, \cdots, \psi_{ni}) \text{ non-}\epsilon\ N \tag{4}$$

for almost all x. Among others, the following results are announced. 1. If the region P defined by $x \in X$,

$$|y_i-y_{i0}|<\int_{x_0}^{x}[M(x)+\gamma]dx$$

for a certain $\gamma>0$ is contained in G, there exists at least one generalized solution. 2. If $f_i(x, y_1, \cdots, y_n)$ is continuous in the y_j for $x \in K_i \subset X$, $y_i=u_i(x)$, then almost everywhere in K_i, $u_i'=f_i(x, u_1, \cdots, u_n)$. 3. The intersection of the set of generalized solutions through a fixed point with a hyperplane $x=$ const. is a continuum. *J. L. Massera.*

Erugin, N. P. Analytic theory of nonlinear systems of ordinary differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 465–486 (1952). (Russian)

This work is related to that of Painlevé, regarding the equation (1) $y''=R(x, y, y')$, where R is a rational function of y' and is algebraic in x, y. The author's approach to the study of singular points is different from that of Painlevé; he considers the question of existence of systems (2) $dx/dz=f_1(x, y, z)$, $dy/dz=f_2(x, y, z)$, not having any m.e.s. (movable, essentially singular) points. Rather wide sufficient conditions on f_1, f_2 are given, under which (2) has no m.e.s. points. For such a system, if $z=z_0$ is a movable singularity, then for $z\to z_0$ one will have $y\to y_0$, $x\to x_0$ (x_0, y_0 possibly infinite). The author then gives a method for the construction of solutions (satisfying initial conditions: $y\to\infty$, $x\to\infty$ for $z\to z_0$), asymptotic to certain series. The author's method is stronger then that of Painlevé, as it enables one to study equations of the form (1), having multi-valued movable singularities, and to study effectively the character of solutions in a neighborhood of a movable singularity. *W. J. Trjitzinsky* (Urbana, Ill.).

Hartman, Philip, and Wintner, Aurel. Linear differential and difference equations with monotone solutions. Amer. J. Math. 75, 731–743 (1953).

The authors' main theorem is the following: Let $A(t)$ be an $n\times n$ matrix of continuous functions satisfying the condition $A(t)\geqq 0$. Then the system of differential equations $x'=-A(t)x$ has at least one solution $x=x(t)$ satisfying $x(t)\geqq 0$, $-x'(t)\geqq 0$. As the authors point out, this is a generalization of a classical theorem of Perron concerning positive matrices. Corresponding results for difference equations are also presented. *R. Bellman.*

Quade, W. Konstruktion einer Integralbasis an einer schwach singulären Stelle. Jber. Deutsch. Math. Verein. 56, 88–104 (1953).

For the nth order linear homogeneous differential equation (*) $\sum_{\nu=0}^{n} z^\nu A_\nu(z) w^{(\nu)}=0$, where $A_n(z)\equiv 1$ and the $A_\nu(z)$ are regular at $z=0$, the author constructs a fundamental set of solutions by introducing in (*) the substitution $z=e^t$ and solving an associated infinite recursive system of linear differential equations with constant coefficients.

W. T. Reid (Evanston, Ill.).

Donskaya, L. I. Construction of a solution of a linear system in the neighborhood of a regular singular point in singular cases. Vestnik Leningrad Univ. **1952**, no. 6, 3–28 (1952). (Russian)

Under consideration is a system in matrix form (1) $X^{(1)}(t)=XP(t)$, where t is complex, $P=\sum_1^\infty P_k t^{-k}$ and the P_k (in canonic form) are matrices of real constants; $t=\infty$ is a regular singular point. Lappo-Danilevskiĭ [Trudy Fis.-Mat. Inst. Steklov. **6** (1934); **7** (1935)] has shown the following: if the c.n.'s (characteristic numbers) of P_1 do not differ by integers, then (1) has the solution (2) $X=t^W Z(t)$ [$W=P_1$; $Z(t)$, $Z^{-1}(t)$ analytic at ∞]; if some of the c.n.'s of P_1 differ by integers, then either there is no solution of the form (2), or there is one and it is not unique. The author considers the problem, suggested by the results of Lappo-Danilevskiĭ, of representing the solution of (1) as a product of two matrices: one comprising completely the singularity of the solution, the other being analytic at ∞. It is shown that, for systems of two and three equations the problem can be solved in the exceptional case in the form (2), where $W\neq P_1$ and $D(Z)\to 0$ with $1/t$; the author also considers cases when a solution of (1) can be obtained in the form $X=t^{P_1}Z(t)$, where Z, Z^{-1} are analytic at ∞.

W. J. Trjitzinsky (Urbana, Ill.).

Siegel, Carl Ludwig. Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 21–30 (1952).

Let differential equations

$$\dot{x}_k=P_k(x_1,\cdots,x_n)\quad(k=1,\cdots,n) \tag{1}$$

be given, in which the P_k are real-analytic at the point $0=(0,\cdots,0)$ and vanish at 0. Let $\lambda_1,\cdots,\lambda_n$ be the characteristic values associated with the equilibrium solution at 0. It is shown that, except for a set of measure 0 of values of $(\lambda_1,\cdots,\lambda_n)$, (1) can be reduced to the normal form $\dot{y}_k=\lambda_k y_k$ $(k=1,\cdots,n)$ by a one-to-one transformation $y_k=F_k(x_1,\cdots,x_n)$, in which the F_k are analytic at 0 and vanish there. This extends classical results, in which the λ_k were assumed to satisfy inequalities [cf. E. Picard, Traité d'analyse, v. III, 3rd. ed., Gauthier-Villars, Paris, 1928, pp. 17–22].

W. Kaplan (Ann Arbor, Mich.).

Mohr, Ernst. Integration von gewöhnlichen Differentialgleichungen mit konstanten Koeffizienten mittels Operatorenrechnung. Math. Nachr. **10**, 1–49 (1953).

The general system of r nonhomogeneous linear differential equations of order n in r unknown functions $Y_i(t)$, $i=1,2,\cdots,r$, with prescribed initial values of each of those functions and of their derivatives up to the order $n-1$, where the coefficients in the differential equations are constants, is treated here. The system is reduced to matrix form and solved by means of the Laplace transformation. Conditions under which the system has a solution and formulas for the solution are arrived at by this process. Degenerate cases are treated, in some cases by replacing the system by a larger system in equations of the first order. The author's purpose was to make a more thorough or exhaustive study of such systems by these methods than has been made heretofore. He presents two detailed numerical examples of his results.

R. V. Churchill (Ann Arbor, Mich.).

Chaundy, T. W. Second-order linear differential equations with polynomial solutions. Quart. J. Math., Oxford Ser. (2) **4**, 81–95 (1953).

W. W. Sawyer [same J. **20**, 22–30 (1949); these Rev. **11**, 31] has investigated ordinary linear differential equations of the second order with polynomial coefficients which contain a parameter k linearly and have the property that for each $n=0,1,2,\cdots$ there is a value k_n of k for which the equation has a polynomial solution of exact degree n. The present author discussed "Sawyer systems" of any order [ibid. **20**, 105–120 (1949); these Rev. **11**, 31] in an attempt to study their reducibility. In the present paper he gives a more detailed analysis of systems of order 2. The equation is put in the form

$$(F-kG)Y=0,\quad F=\sum_{r=0}^{p}x^r f_r(\delta),\quad G=\sum_{r=0}^{p}x^r g_r(\delta),$$

where $\delta\equiv x(d/dx)$, f_r and g_r are quadratic polynomials in δ, p is called the rank of the equation, and it is assumed that by a fractional linear change of k the equation has been written in a form in which neither $k=0$ nor $k=\infty$ is one of the special values k_n.

The reducibility of the equation depends on the power series solutions of the equation $G(x,\delta)y=0$. The case when this equation has no solution which is an infinite power series was adequately discussed in §4.5 of the author's previous paper, the result being that in this case the system can be reduced to one of rank zero. The case when $Gy=0$ has exactly one infinite series solution is shown to be reducible to an equation of rank one (i.e., to a hypergeometric equation). For equations of rank two, with $f_0(\delta)=g_0(\delta)=\delta(\delta-m)$, the author proves that if $(F-kG)Y=0$ has polynomial solutions Y_n for $0<n<m$, with $Y_m\neq x^m$, then it has polynomial solutions of all degrees, and, moreover, the series solutions of the equation for arbitrary k are free of logarithms. The assertion of the earlier paper that every Sawyer system is reducible to the type $(\mathfrak{A})$ or $(\mathfrak{A}_m)$ cannot be maintained, but it can be shown that equations of rank two, with $f_0=g_0=\delta(\delta-m)$, are Sawyer systems if and only if there is an operator $P(x,\delta)$ so that PF and PG are both of the type

$$A_m=\delta\cdots(\delta-m)+\sum_{r=1}^{m}x^r\delta\cdots(\delta-m+r)h_r(\delta)+x^{m+1}h_{m+1}(\delta)$$

where $h_r(\delta)$ is a polynomial of degree r with constant coefficients. $P(x,\delta)$ is itself of the form A_{m-1} with $h_r(\delta)$ constants and $h_{m+1}\neq 0$.

A. Erdélyi (Pasadena, Calif.).

Ważewski, T. Sur l'évaluation du nombre des paramètres éssentiels dont dépend la famille des intégrales d'un système d'équations différentielles ayant une propriété asymptotique. Bull. Acad. Polon. Sci. Cl. III. **1**, 3–5 (1953).

Sia $X'=f(t,X,Y)$, $Y'=g(t,X,Y)$ un sistema di equazioni differenziali ordinarie scritto in forma vettoriale, con X e Y vettori a p e q dimensioni rispettivamente, i secondi membri del sistema essendo continui in un insieme aperto W contenuto fra gli iperpiani $t=a$ e $t=b$ $(a<b)$; e per ogni punto di W passi una sola soluzione del sistema. Siano $L(t)$ ed $M(t)$ funzioni vettoriali, continue insieme con le loro derivate, al pari delle funzioni scalari e positive $l(t)$ ed $m(t)$, nell'intervallo aperto di estremi a e b; e l'insieme T definito dalle limitazioni $|X-L(t)|\leq l(t)$, $|Y-M(t)|\leq m(t)$, $a<t<b$

appartenga a W. Si supponga poi che sia

$$\{X-L(t)\}\{f(t, X, Y)-L'(t)\}<l(t)l'(t)$$

se $|X-L(t)|=l(t)$, $|Y-M(t)|\leqq m(t)$, $a<t<b$ e che sia

$$\{Y-M(t)\}\{g(t, X, Y)-M'(t)\}>m(t)m'(t)$$

se $|X-L(t)|\leqq l(t)$, $|Y-M(t)|=m(t)$, $a<t<b$. Allora l'insieme Z dei punti P, contenuti nella sezione di T coll'iperpiano $t=t_0$, siffatti che la soluzione del sistema $X'=f$, $Y'=g$ passante per P appartenga a T almeno finché t è maggiore di t_0, ha una dimensione non inferiore a p. L'A. dà un cenno della dimostrazione. *G. Scorza Dragoni.*

Mikołajska, Z. Sur les mouvements asymptotiques d'un point matériel mobile dans le champ des forces repoussantes. Bull. Acad. Polon. Sci. Cl. III. 1, 11–13 (1953).

L'equazione $\ddot{X}=F(t, X, \dot{X})$, fra vettori a n componenti, ammette una famiglia di soluzioni decrescenti in senso largo a partire da un certo istante in poi e dipendenti da n parametri essenziali, se $F(t, X, \dot{X})$ è continua, se le soluzioni della $\ddot{X}=F$ sono individuate dai valori iniziali e se il prodotto scalare di X e di F non è mai negativo.

G. Scorza Dragoni (Padova).

Zlámal, Miloš. Asymptotic properties of the solutions of the third order linear differential equations. Publ. Fac. Sci. Univ. Masaryk **1951**, 159–167 (1951). (Russian summary)

The properties of the solutions of $y'''+p(x)y'+q(x)y=0$, as $x\to+\infty$, are investigated under various hypotheses on the coefficients. In all cases $p'(x)$ and $q(x)$ are assumed continuous. The following is an incomplete summary of the results proved by the author. If $p(x)\geqq 0$, $\limsup p(x)x^{-1/2}<\infty$, $\limsup q(x)x^{1/2}<\infty$, and $p'(x)-2q(x)\geqq d>0$, then every non-trivial solution is either oscillatory or diverges to $\pm\infty$, and the zeros of any two independent oscillatory solutions interlace. The oscillatory solutions are of class L_2. If $p(x)\geqq m>0$, $q(x)\geqq m$, $q(x)-p(x)\geqq 0$, then every solution is either oscillatory or tends to zero together with its first derivative, and the non-oscillatory solutions are of class L_2. The author's main tool is an identity due to Mammana [Math. Z. **33**, 186–231 (1931)]. *W. Wasow.*

Jeffreys, Harold. On approximate solutions of linear differential equations. Proc. Cambridge Philos. Soc. **49**, 601–611 (1953).

The author discusses a technique for finding asymptotic expressions, when h is large, for the solutions of differential equations of the type $d^2y/dx^2=(h^2\chi_0(x)+h\chi_1(x)+\chi_2(x))y$. The emphasis is on approximations for which the relative error is uniformly $O(1/h)$ in large complex x-domains E, particularly in unbounded ones. If $\chi_0(0)\neq 0$ this is achieved by reducing the differential equation to the form $d^2z/d\xi^2-h^2z=g(\xi, h)z$ by means of a transformation of the form

$$\xi=\int_0^x(\chi_0+\chi_1/h+\psi_2/h^2)^{1/2}dx, \quad y=(d\xi/dx)^{-1/2}z,$$

where $\psi_2(x)$ must be chosen so that $g(\xi, h)$ satisfies certain stated conditions in E. If these conditions are satisfied, a fundamental system $z=Z_j(x, h)$, $j=1, 2$, of the transformed differential equation can be shown—by means of a transformation into a Volterra integral equation—to be approximately equal to $e^{\pm h\xi}$, in E. If $\chi_0(0)=0$, but $\chi_1(0)\neq 0$, a transformation defined by $\xi^{3/2}\xi=\chi_0+\chi_1/h+\psi_2/h^2$ may be used, which permits, in analogous fashion, the approximation of the solutions by those of Airy's equation. The paper does not contain a general method for the calculation of a function $\psi_2(x)$ for which $g(\xi, h)$ has the required properties.

W. Wasow (Los Angeles, Calif.).

***Borg, Göran. Deux notes concernant la stabilité. Two notes concerning stability.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 21–29, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and English)

These closely connected, brief, and not entirely clear notes relate to the following problem. In the differential equation $d^2Y/d\xi^2+LY=F(t)Y$, Y is a one-column matrix with n elements, L is a diagonal matrix with positive constant elements (the square roots of no two of them being congruent mod 1), and $F(t)$ is a square matrix with elements which are continuous periodic functions with the period 2π. If $F(t)\equiv 0$, all solutions are bounded. It is desired to determine "how large" $F(t)$ can become before unbounded solutions appear. In the first and principal note a quadratic form is constructed, having the elements of $F(t)$ as coefficients, and a sufficient condition for the boundedness of the solutions is stated in terms of a certain upper bound for this form. The second note is devoted to a few remarks on another approach to the same problem. Although the details are by no means clear, the ideas discussed are interesting, and a more adequate exposition of them would be welcome.

L. A. MacColl (New York, N. Y.).

***Haag, Jules. Les mouvements vibratoires.** Presses Universitaires de France, Paris, 1952. 268 pp. 1,600 francs.

This is a carefully written exposition of the theory of vibrations, which begins with the rudiments of the subject and terminates with some discussion of the vibrations of non-linear systems having one degree of freedom. The author indicates that systems having several degrees of freedom will be treated in another volume.

Chapters I and II give a thorough discussion of the kinematical properties of sinusoidal motions and motions that are resultants of sinusoidal motions. Chapters III and IV deal with the ordinary theory of the linear oscillator, and Chapter V deals with the linear oscillator subjected to a perturbing force of a very general kind. The next three chapters constitute the most noteworthy part of the book. Here the author considers what he calls the problem of synchronization, by which is meant the problem of maintaining an oscillator in a state of periodic motion, with a prescribed period, in the presence of various perturbing forces. The choice of topics is strongly influenced by the author's professional interest in chronometry, and the wealth of detail makes this part of the book rather laborious reading. The ninth chapter, comprising 33 pages, deals with non-linear oscillators. The final chapter is devoted to some special problems drawn from the field of chronometry.

Except for the fact that the chapter on non-linear oscillators seems too brief, the reviewer considers this to be an excellent treatment of the subject. Only a few trivial errors were noticed. The book has a pleasing appearance.

L. A. MacColl (New York, N. Y.).

***Amerio, Luigi. Sur l'extension de quelques points de la théorie de Poincaré aux systèmes de deux équations différentielles à trois variables.** Actes du Colloque International des Vibrations non linéaires. Ile de Porquerolles, 1951, pp. 261–269; discussion, p. 268, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953).

This is mainly a restatement of results which the author has already published elsewhere [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 206–212, 289–297 (1951); these Rev. **12**, 827; **13**, 346]. A few additional results are indicated briefly. *L. A. MacColl.*

Antosiewicz, H. A. Correction to the paper "Forced periodic solutions of systems of differential equations". Ann. of Math. (2) **58**, 592 (1953).

See same Ann. (2) **57**, 314–317 (1953); these Rev. **14**, 751.

Karimov, D. H. On periodic solutions of nonlinear equations of the fourth order. Doklady Akad. Nauk UzSSR **1949**, no. 8, 3–7 (1949). (Russian. Uzbek summary)

See Doklady Akad. Nauk SSSR (N.S.) **57**, 651–653 (1947); these Rev. **9**, 240.

Abramowitz, Milton. On the solution of the differential equation occurring in the problem of heat convection in laminar flow through a tube. J. Math. Physics **32**, 184–187 (1953).

The lower characteristic values of the system

$$d^2y/dr^2+(1/r)dy/dr+\beta^2(1-r^2)y=0, \quad y(0)=1, \quad y(1)=0,$$

are calculated as follows. The author expands $y(r, \beta)$ in a series involving Bessel functions of integer order and determines $y(1, \beta)$ for a number of values in the vicinity of a characteristic value. By inverse interpolation the required characteristic values are found. The higher characteristic values can be determined from the asymptotic expansion obtained by Lauwerier [Appl. Sci. Research A. **2**, 184–204 (1950); these Rev. **12**, 333]. For the third and fourth characteristic value both methods are well in agreement.

H. A. Lauwerier (Amsterdam).

Hölder, E. Über den Aufbau eines erweiterten Greenschen Tensors kanonischer Differentialgleichungen aus assoziierten Lösungssystemen. Ann. Soc. Polon. Math. **25** (1952), 115–121 (1953).

The author determines a symmetric generalized Green's matrix for a compatible two-point boundary-value problem of the type that is given by the canonical form of the accessory problem for a calculus of variations problem with separated end-conditions. *W. T. Reid* (Evanston, Ill.).

Conti, Roberto. Su una classe generale di problemi ai limiti non lineari per i sistemi di due equazioni differenziali ordinarie del primo ordine. Rend. Sem. Mat. Univ. Padova **22**, 181–191 (1953).

This paper is concerned with a differential system of the form

$$(*) \qquad y'=F(x, y, z)z+\phi_1(x, y, z), \quad z'=\phi_2(x, y, z),$$

where F, ϕ_1, ϕ_2 are real-valued functions defined in the region $x_0 \leqq x$, $|y|<\infty$, $|z|<\infty$. Under conditions too complicated to detail here the author shows that (*) possesses at least one solution joining the specified line γ_1: $x=x_0$, $y=y_0$ and the specified curve γ_2: $y=Y(x)$, $z=Z(x)$.

W. T. Reid (Evanston, Ill.).

Molčanov, A. M. On conditions for discreteness of the spectrum of self-adjoint differential equations of the second order. Trudy Moskov. Mat. Obšč. **2**, 169–199 (1953). (Russian)

The author considers the question of discreteness or non-discreteness of the spectrum of the eigenvalue problem concerning the equation (1) $-\Delta u+pu=\lambda u$, where the function p is supposed uniformly bounded below in the whole n-dimensional space $\mathfrak{E}^n$. The author first considers the problem in the whole of $\mathfrak{E}^n$ and obtains a complete answer to the question which can be stated as follows. I. a) For the non-discreteness of the spectrum it is necessary that for every $\epsilon>0$ there exist a sequence $\{\mathfrak{D}_k\}$ of disjoint cubes of equal side length, and a sequence $\{\mathfrak{F}_k\}$ of closed subsets of $\{\mathfrak{D}_k\}$ with $c_n(\mathfrak{F}_k) \leqq \epsilon c_n(\mathfrak{D}_k)$ ($c_n(\mathfrak{A})$ denotes the capacity of $\mathfrak{A}$ in $\mathfrak{E}^n$) such that $p_k=\int_{\mathfrak{D}_k-\mathfrak{F}_k} p\,dx$ is bounded for $k\to\infty$. b) It is sufficient for the non-discreteness of the spectrum that the sequences $\{\mathfrak{D}_k\}$ and $\{\mathfrak{F}_k\}$ satisfying the above conditions exist for at least one ϵ smaller than a constant ϵ_0, ϵ_0 depending only on the dimension n.

An evaluation of ϵ_0 is given which is certainly not the best possible. In the one-dimensional case the "exceptional" sets $\mathfrak{F}_k$ and the notion of capacity are superfluous and Statement I takes the simpler form: I'. A necessary and sufficient condition for the non-discreteness of the spectrum is the existence of a disjoint infinite sequence of intervals $\mathfrak{D}_k$ of equal lengths such that $\int_{\mathfrak{D}_k} p\,dx$ is bounded.

The author then applies his methods and results to the case of equation (2) $-\Delta u=\lambda u$ ($p\equiv 0$) in a non-bounded domain $\mathfrak{G}$ of $\mathfrak{E}^n$ with the boundary condition $u=0$ on the boundary Σ of $\mathfrak{G}$. His main result can be stated as follows. II. A necessary and sufficient condition for the non-discreteness of the spectrum is the existence of a sequence $\{\mathfrak{D}_k\}$ of disjoint cubes of equal side lengths such that $c_n(\mathfrak{D}_k-\mathfrak{G}) \leqq \epsilon_0 c_n(\mathfrak{D}_k)$ where ϵ_0 has the same significance as in Ib.

For special kinds of domains $\mathfrak{G}$ the condition becomes much simpler: II'. The existence of an infinite sequence $\{\mathfrak{D}_k\}$ of disjoint cubes contained in $\mathfrak{G}$ is necessary and sufficient for the non-discreteness of the spectrum when 1) the dimension $n=2$, and $\mathfrak{G}$ is simply connected; 2) the dimension $n \geqq 3$ and the boundary Σ is connected, smooth, and of uniformly bounded curvature.

In this paper significant progress is made in the treatment of a classical problem which started with the work of H. Weyl and has been continued more recently in the work of Titchmarsh, Friedrichs, and Rellich. In view of the interest of the results, the reviewer feels that the following comments and corrections are indicated.

For Statement I, the starting idea is to divide the whole space into a net of cubes of equal side lengths D. The union of spectra (with multiplicities counted) for equation (1) for all the cubes with free boundary condition ($du/dn=0$) then minorates the spectrum in the whole space, whereas the union of spectra in all cubes with fixed boundary condition ($u=0$) majorates it. This idea is not new; it was the basis of classical results of H. Weyl and R. Courant concerning the asymptotic distribution of eigenvalues. The essential tools which allow the author to obtain his general statements are the use of capacities of sets and the following lemma. Lemma. Let μ_0 and λ_0 be the smallest eigenvalues of equation (1) in a cube of side length D with the boundary condition $du/dn=0$ or $u=0$ respectively. If $p \geqq 0$ in the cube, then there exist two constants A and B depending only on the dimension n (and not on p!) such that $\mu_0 \leqq \lambda_0 \leqq A\mu_0+B/D^2$.

In the paper the values of the constants A and B ($A=B=2^{2n+7}$) are not the best possible; if it were possible to choose the constant $A=1$, the results of the paper could be extended also to p non-bounded. The restriction that p be uniformly bounded below is rather a disadvantage in view of the intention of the author to apply his results to problems in quantum mechanics where p is usually non-bounded below. In this respect the recent results of Friedrichs [see Comm. Pure Appl. Math. 3, 439–449 (1950); these Rev. 13, 133] have the advantage of allowing one to treat some cases where p is unbounded below. However, under the limitations imposed by the author, in particular for equation (2), the results of the paper give as immediate consequences all previously known results [for example, F. Rellich, Courant Anniversary Volume, Interscience, New York, 1948, pp. 329–344; these Rev. 9, 355]. It may be added here that the simplified necessary and sufficient conditions are valid for much larger classes of domains than is explicitly stated by the author, in particular, for $n=2$, it is enough that the components of the complement of $\mathfrak{G}$ have diameters $\geqq\alpha$ for some $\alpha>0$.

In his development the author is rather vague on several points which may lead to misunderstandings. These may be clarified as follows.

1. In introducing the capacity c_n in n-dimensional space, the author does not seem to realize that the case $n=2$ is an exceptional one because there is no suitable notion of capacity with respect to the whole plane as there is one for every $\mathfrak{E}^n$ with $n\geqq 3$. This can be easily remedied by considering in Statements I and II that the capacities $c_2(\mathfrak{F}_k)$ and $c_2(\mathfrak{D}_k)$ are actually the logarithmic capacities of $\mathfrak{F}_k$ and $\mathfrak{D}_k$ with respect to a circle C_k concentric with $\mathfrak{D}_k$ and with radius equal to twice the side length D_k.

2. In passing to the consideration of the spectrum in a domain the author makes two general statements which may be misleading: a) he says that his methods are applicable and similar results can be obtained for the case of any self-adjoint boundary conditions imposed on the boundary Σ. However, it seems to the reviewer that this is true only for the boundary condition $u=0$; that for other boundary conditions one could not expect similar results is shown already by the fact that there are bounded domains in the plane with very smooth boundary except at one boundary point for which the spectrum of equation (2) with boundary condition $du/dn=0$ is non-discrete. b) The author suggests that the problem in a domain $\mathfrak{G}$ with boundary condition $u=0$ can be transformed into a problem in the whole of $\mathfrak{E}^n$ (so that Statement I is applicable) by putting $p=+\infty$ in the complement of $\mathfrak{G}$. However, a direct application of this idea is not always feasible due to the possibility that some parts of Σ may belong to the interior of the closure $\bar{\mathfrak{G}}$ (e.g. in the case of a slit). However, the idea can be applied and Statement I becomes valid for the spectrum in $\mathfrak{G}$ if the condition $\mathfrak{F}_k\supset\mathfrak{D}_k-\mathfrak{G}$ is imposed additionally on the exceptional sets $\mathfrak{F}_k$. Statement II is obtained in precisely this way from Statement I. *N. Aronszajn.*

Maass, Hans. Über die Zurückführung der Eigenwertprobleme bei gewöhnlichen Differentialgleichungen auf Integralgleichungen. Math. Z. 58, 385–390 (1953).

A self-adjoint boundary problem of the form

$$\sum_{\nu=0}^{n}(-1)^\nu[f_\nu(x)y^{(\nu)}(x)]^{(\nu)}=\lambda\sum_{\nu=0}^{n}(-1)^\nu[g_\nu(x)y^{(\nu)}(x)]^{(\nu)},$$

$$U_\mu[y]\equiv\sum_{\nu=0}^{2m-1}[\alpha_{\mu\nu}y^{(\nu)}(a)+\beta_{\mu\nu}y^{(\nu)}(b)]=0\quad(\mu=1,\cdots,2m)$$

with $f_n(x)\neq 0$ and $f_\nu(x)$, $g_\nu(x)$ of class $C^{(\nu)}$ on $a\leqq x\leqq b$, and for which $\lambda=0$ is not a proper value, is shown to be equivalent to a system of homogeneous integral equations of the second kind with a symmetric kernel matrix in case: (i) $n<m$; (ii) for each ν ($\nu=1,\cdots,n$) either $g_\nu(x)\equiv 0$ on $a\leqq x\leqq b$ or $g_\nu(x)>0$ on this interval; (iii) if u and v satisfy the boundary conditions $U_\mu=0$, then

$$\sum_{\nu=0}^{n}\sum_{\mu=0}^{\nu-1}(-1)^{\nu-\mu}u^{(\mu)}(x)[g_\nu(x)v^{(\nu)}(x)]^{(\nu-\mu-1)}\Bigg|_{x=a}^{x=b}=0.$$

W. T. Reid (Evanston, Ill.).

Pini, Bruno. Sui cicli relativi ai sistemi ai differenziali totali. Rend. Sem. Mat. Univ. Padova 22, 38–63 (1953).

The Pfaffian system $dx=a(x)du+b(x)dv$, where x, a, b are vectors and u, v scalars in 3-space is assumed passive. Continuing a previous study [Ann. Mat. Pura Appl. (4) 34, 95–104 (1953); these Rev. 14, 1088] this paper now considers curves which are characteristic, closed and without singularity. The results are analogous to those of Poincaré [Oeuvres, t. 1, Gauthier-Villars, Paris, 1928, pp. 137–158] for ordinary systems but are established independently.

J. M. Thomas (Durham, N. C.).

***Petrovskij, I. G. Parciální diferenciální rovnice.** [Partial differential equations.] Přírodovědecké Vydavatelství, Praha, 1952. 276 pp. 180 Kčs.

Translation by J. Kurzweil of "Lekcii ob uravneniyah s častnymi proizvodnymi" [Gostehizdat, Moscow-Leningrad, 1950; these Rev. 13, 241].

RUSSIAN on order

Loewner, Charles. Generation of solutions of systems of partial differential equations by composition of infinitesimal Baecklund transformations. J. Analyse Math. 2, 219–242 (1953).

Der an der reellen Achse gespiegelte Geschwindigkeitsvektor $w=u-iv$ einer ebenen stationären reibungs- und wirbellosen inkompressiblen Flüssigkeit hängt analytisch von der Stelle $z=x+iy$ ab. Damit ergeben sich die Cauchy-Riemannschen Differentialgleichungen als ein Spezialfall der Matrizengleichung

$$(*)\qquad w_x=Hw_y,\quad w=\begin{pmatrix}u\\-v\end{pmatrix},\quad H=\begin{pmatrix}0,&1\\-1,&0\end{pmatrix}.$$

Sie bedeuten eine Verknüpfung der konjugierten Potentialfunktionen $u(x,y)$ und $v(x,y)$ derart, dass, wenn u bekannt ist, auch v bis auf Integrationskonstante bekannt ist und umgekehrt. u wie v genügen dann bekanntlich derselben (Laplaceschen) Differentialgleichung zweiter Ordnung. Eine erste Verallgemeinerung der Matrizengleichung (*) ergibt sich durch Wahl einer beliebigen nichtsingulären Matrix $H=\begin{pmatrix}h_{11}&h_{12}\\h_{21}&h_{22}\end{pmatrix}$ in der Form

$$(**)\qquad \zeta_x=H\zeta_y,\quad \zeta=\begin{pmatrix}\xi\\\eta\end{pmatrix}.$$

Auch (**) bedeutet eine Verknüpfung zweier jetzt assoziiert genannter Funktionen $\xi(u,v)$ und $\eta(u,v)$, die aber jetzt jede für sich nicht mehr einer und derselben Differentialgleichung genügen. Wiederum ist η durch einfache Integrationsprozesse zu ermitteln, wenn ξ bekannt ist und umgekehrt. Wenn man nun aber an Stelle von je einer Funktion ξ und η je ein Paar solcher

$$\zeta=\begin{pmatrix}\xi\\\eta\end{pmatrix},\quad \zeta'=\begin{pmatrix}\xi'\\\eta'\end{pmatrix}$$

in Betracht zieht? Dann handelt es sich um assoziierte Matrizen ζ und ζ' mit Bezug auf ein System von Differentialgleichungen, das Verfasser in einer vorhergehenden Untersuchung [NACA Tech. Note no. 2065 (1950); diese Rev. 13, 464] in der Form

$$(***)\qquad \begin{aligned}\zeta_u' &= W\zeta_u + A\zeta + C\zeta'\\ \zeta_v' &= W\zeta_v + B\zeta + D\zeta'\end{aligned}$$

aufgestellt hat. Die Koeffizienten A, B, C, D und W der Differentialgleichungen (***), welche eine Bäcklundsche Transformation definieren, sind quadratische Matrizen von Grad 2, deren Elemente Funktionen von u and v sind. Für ein Grundgebiet Δ der u, v wird insbesondere W nichtsingulär vorausgesetzt. Geht man dann von einer nichtsingulären Matrix H' aus, so kann immer $H=W^{-1}H'W$ gebildet werden. Das System (***) kann jetzt als eine (weitere) vektorielle Verallgemeinerung des Cauchy-Riemannschen Systems aufgefasst werden, indem es die Verknüpfung der Matrizen ζ und ζ', also die von 2 Paaren von Funktionen, beschreibt, wie jenes die zwischen einer harmonischen (Einzel) Funktion und ihrer Konjugierten. Gibt man die Matrix ζ vor, so folgt aus (***) eine Integrabilitätsbedingung für ζ' und diese verwandelt sich in eine Bedingung für ζ, mit assoziierten Matrizen ζ' behaftet zu sein, wenn die Integrabilitätsbedingung selbst noch von ζ' befreit wird, was durch zusätzliches Nullsetzen des Faktors der Matrix ζ' in der Integrabilitätsbedingung erreicht wird. Weitere Vereinfachungen ergeben sich im Falle, dass die Differentialgleichungen ζ nicht explizit enthalten.—Das nächste Ziel ist nun, Garantieen dafür zu finden, dass jede Lösung der Matrizendifferentialgleichung

$$\zeta_u - H\zeta_v = 0, \quad H = W^{-1}H'W$$

durch eine passende Bäcklundtransformation (***) mit einer Matrix ζ' assoziiert werden kann, die ihrerseits Lösung der Matrizendifferentialgleichung

$$\zeta_u' - H'\zeta_v' = 0$$

(mit vorgeschriebener Matrix H') ist. Diese Garantieen werden in Form von 5 Bedingungen für die 5 Matrizen der Bäcklund Transformation angegeben (Satz 1). Nun seien $\sigma_1, \sigma_2, \cdots, \sigma_n$ gegebene Bäcklundsche Transformationen und $\zeta_u^{i} = H^i\zeta_v^{i}$ $n+1$ Matrizengleichungen derart, dass σ_i die $i$te dieser Gleichungen in die $(i+1)$te transformiert $(i=1, 2, \cdots, n)$. Man erhält auf diese Weise von irgendeiner Lösung von $\zeta_u' = H'\zeta_v'$ im allgemeinen eine $2n$-parametrige Schar von Lösungen der letzten Gleichung $\zeta_u^{n+1} = H^{n+1}\zeta_v^{n+1}$. Dazu sind hintereinander n Systeme gewöhnlicher Differentialgleichungen zu integrieren. Im Sinne eines Grenzüberganges $n\to\infty$ wird man jetzt auf das Problem der Zusammensetzung infinitesimaler Bäcklundscher Transformationen geführt. Die Bäcklundsche Transformation (***) kann zunächst folgendermassen geschrieben werden:

$$(\Delta\zeta)_u = (W-I)\zeta_u + (A+C)\zeta + C\Delta\zeta,$$

$$\Delta\zeta = \zeta' - \zeta, \quad I = \begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix}.$$

$$(\Delta\zeta)_v = (W-I)\zeta_v + (B+D)\zeta + D\Delta\zeta,$$

Nun seien die Elemente aller Matrizen Funktionen nicht nur von u und v sondern auch solche eines Parameters t im Intervall $t_0\leqq t\leqq t_1$. Schreibt man jetzt partielle Ableitungen ζ_{ut}, ζ_{vt} und ζ_t an Stelle der Differenzen $(\Delta\zeta)_u$, $(\Delta\zeta)_v$ und $\Delta\zeta$ und A, B wieder für $A+C$ und $B+D$, so entsteht das System

$$\begin{aligned}\zeta_{ut} &= \Omega\zeta_u + A\zeta + C\zeta_t\\ \zeta_{vt} &= \Omega\zeta_v + B\zeta + D\zeta_t\end{aligned} \qquad \Omega = W - I.$$

Eine Lösung $\zeta(u, v, t_1)$ dieses Systems entsteht aus einer solchen $\zeta(u, v, t_0)$ durch Zusammensetzen solcher infinitesimaler Transformationen. Die Kardinalfrage ist jetzt: unter welchen Bedingungen für die Matrizenkoeffizienten in den Relationen der infinitesimalen Transformation gibt es eine Matrix $H(u, v, t)$ $(t_0\leqq t\leqq t_1)$ derart, dass irgend eine Lösung der Differentialgleichung

$$\zeta_u = H(u, v, t)\zeta_v$$

für einen speziellen Wert t' des Intervalls $t_0\leqq t'\leqq t_1$ zu einer Lösung $\zeta(u, v, t)$ für alle t des Intervalls fortgesetzt werden kann? Kann dabei $\zeta(u_0, v_0, t)$ im Grundbereich beliebig vorgeschrieben werden? Diese Bedingungen werden in Satz 2 explizit angegeben und bewiesen. Die Bedingungen dieses Satzes werden jetzt weiterhin untersucht und gedeutet. Das geschieht vorerst ohne weitere einschränkende Annahmen. Für aerodynamische Zwecke entspricht jedoch meistens die Spezialisierung

$$H = \begin{pmatrix}0, & h\\ -1/h, & 0\end{pmatrix}, \quad h = h(u, v, t)$$

dem Sachverhalt. In dieser Vereinfachung führt jedoch die Untersuchung auf die Klasse hyperbolischer Differentialgleichungen

$$g_{tu} = a_1e^g + a_2e^{-g}, \quad a_1 = a_1(t), \quad a_2 = a_2(t)$$

derart, dass irgend eine Lösung dieser hyperbolischen Differentialgleichung die Möglichkeit vermittelt, aerodynamisch wichtige Differentialgleichungssysteme untereinander zu transformieren, in der Hoffnung auf diese Weise auf einfachere Verhältnisse zu stossen. Insbesondere werden dabei jene Systeme von Interesse, die sich in das System der Cauchy-Riemannschen Differentialgleichungen transformieren lassen. Auch die bekannte Deutung der Integralflächen der kompressiblen Geschwindigkeitspotentialgleichung als Relativminimalflächen im E. Müllerschen Sinne scheint auf diesem Wege einer Auswertung zugänglicher zu sein. Die eben erwähnte hyperbolische Differentialgleichung kann auf drei kanonische Fälle reduziert werden:

(1) $g_{tu} = \frac{1}{2}(e^g + e^{-g}) = \sinh g$; (2) $g_{tu} = \frac{1}{2}(e^g - e^{-g}) = \cosh g$;

(3) $g_{tu} = e^g$.

Für Transformationen in das Cauchy-Riemannsche System empfiehlt sich (1). Weitere Transformation der Parameter u und t ermöglicht im Falle (1) einen verallgemeinerten Potenzreihenansatz für die unbekannte Funktion g. Mit Konvergenzuntersuchungen und Majorantenbestimmungen dieser Entwicklung schliesst diese bedeutsame Untersuchung. *M. Pinl* (Dacca).

Titt, E. W., McCulley, W. S., Donaldson, Fletcher W., Osborn, Roger, Worthington, L. G., and Long, W. C. Vector algebras and potentials. J. Rational Mech. Anal. 2, 413–442 (1953).

Ce travail présente un formalisme algébrique préliminaire à la classification des équations aux dérivées partielles linéaires et à coefficients constants du second ordre

$$(*)\qquad A^{ij}(\partial^2u/\partial x^i\partial x^j) + B^i(\partial u/\partial x^i) + Cu = f(x^i), \qquad A^{ij} = A^{ji}, \quad i, j = 1, \cdots, n,$$

selon la théorie des caractéristiques. Il contient en outre une étude détaillée de deux solutions particulières de ces équations. L'une, le potentiel de Green, s'identifie à la solution élémentaire de J. Hadamard. L'autre est appelée potentiel retardé. Pour une équation (*) hyperbolique, elle dépend (à un facteur d'amortissement près), non seulement de la

distance non-euclidienne $A_{ij}x^ix^j$, comme la précédente, mais aussi de la projection non-euclidienne du vecteur x^i sur un plan non-caractéristique passant par l'origine. Si l'équation est parabolique, les deux potentiels dépendent de l'analogue de cette projection et de la longueur non-euclidienne du segment projetant. Dans ce dernier cas, on les obtient par un passage à la limite, en partant des potentiels correspondants relatifs à l'équation hyperbolique. Pour $n=3$, les potentiels de Green et retardé coïncident et les auteurs déterminent alors une solution particulière complémentaire de l'équation (*). Les potentiels retardés paraissent n'avoir été rencontré jusqu'ici que dans des cas très particuliers [cf., pour $n=3$, V. Volterra, Leçons sur l'intégration des équations différentielles aux dérivées partielles, Hermann, Paris, 1912]. *H. G. Garnir* (Liège).

McCulley, W. S., and Titt, E. W. Integration formulae and boundary conditions for the hyperbolic equation with three independent variables and regions interior to the cone. J. Rational Mech. Anal. **2**, 443–484 (1953).

Les auteurs exposent une nouvelle méthode d'intégration des problèmes de Cauchy et aux limites relatifs à l'équation à coefficients constants

$$A^{ij}(\partial^2 u/\partial x^i\partial x^j)+B^i(\partial u/\partial x^i)+Cu=f(x^i),$$
$$A^{ij}=A^{ji},\quad i,j=1,2,3,$$

du type hyperbolique (dtm $A^{ij}\neq 0$, $A_{ij}x^ix^j=0$ pour un vecteur x^i).

Le rôle essentiel est joué par un facteur intégrant, solution de l'équation adjointe, nul avec ses dérivées premières sur le cône caractéristique et caractérisé par une discontinuité par saut brusque de sa dérivée normale non-euclidienne au travers d'un plan auxiliaire fixe, π, passant par le sommet du cône caractéristique et le coupant selon deux génératrices réelles. Les auteurs construisent ce facteur intégrant en partant de la solution élémentaire de J. Hadamard, relative au point ξ, intégrée sur la section par le plan π de la figure formée par le cône caractéristique direct issu de ξ et le cône caractéristique rétrograde ayant pour sommet un point x, intérieur au cône précédent.

Grâce à ce facteur intégrant, les auteurs établissent différents types de formules exprimant la valeur de u en un point, en fonction d'expressions intégrales portant sur le domaine limité par la nappe du cône caractéristique rétrograde ayant ce point pour sommet et une surface arbitraire suffisamment régulière, d'orientation d'espace ou non, ou sur la partie de cette surface contenue dans le cône. Les calculs sont effectués grâce à des règles de Leibnitz, généralisées aux intégrales multiples dans les cas où la règle classique conduit à des intégrales divergentes [cf. E. W. Titt, Bull. Amer. Math. Soc. **57**, 276–277 (1951)]. Les formules obtenues peuvent être utilisées pour résoudre des problèmes de Cauchy ou aux limites. Comme exemple, les auteurs traitent d'une manière détaillée le problème aux limites posés pour l'équation des ondes,

$$-c^2(\partial^2 u/\partial x^2)-c^2(\partial^2 u/\partial y^2)+\partial^2 u/\partial t^2=f,$$

avec les données initiales sur $t=0$ et la donnée de Dirichlet sur $y=0$.

La présente méthode d'intégration s'apparente à celle de V. Volterra [Leçons sur l'intégration des équations différentielles aux dérivées partielles, Hermann, Paris, 1912]. Elle conduit à substituer à la discontinuité infinie de la solution élémentaire de J. Hadamard une discontinuité par saut brusque au travers du plan π, au lieu de la remplacer par une discontinuité infinie sur l'axe du cône caractéristique. Comme la méthode de Volterra, elle tombe sous le coup des critiques de J. Hadamard [Le problème de Cauchy, Hermann, Paris, 1932; cf. spéc. §§43–45]. *H. G. Garnir* (Liège).

Orloff, Constantin P. Recherche de l'intégrale générale des équations différentielles partielles du second ordre, qui ne sont pas Monge-Ampériennes. Bull. Acad. Serbe Sci. (N.S.) **5**, Cl. Sci. Math. Nat. Sci. Math. **1**, 37–88 (1952).

This is an almost literal translation of a paper in Serbian [Srpska Akad. Nauka. Posebna Izdanja **142**. Prirodnjački i Matematički Spisi **41**. Belgrade, 1948; these Rev. **11**, 181]. *W. Feller* (Princeton, N. J.).

Salehov, G. S., and Fridlender, V. R. On a problem inverse to the Cauchy-Kovalevskaya problem. Uspehi Matem. Nauk (N.S.) **7**, no. 5(51), 169–192 (1952). (Russian)

Consider, for definiteness, the partial differential equation

$$(1)\quad \frac{\partial^p u}{\partial t^p}=F\left(t, x_1, \cdots, x_n, u, \cdots, \frac{\partial^q u}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}, \cdots\right)$$

with initial Cauchy data

$$(2)\quad \left.\frac{\partial^k u}{\partial t^k}\right|_{t=0}=\varphi_k(x_1, \cdots, x_n),\quad k=0, 1, \cdots, p-1.$$

The equation (1) is called normal (following S. Kovalevsky) provided that $\alpha_0<p$, $q\leq p$, and otherwise is called anormal. The subject matter of the present survey article may be properly said to begin with the Cauchy-Kovalevsky theorem, which asserts that a normal equation (1) has a solution, analytic in a neighborhood of $t=0$, for arbitrary analytic Cauchy data (2). In general, this is not true for anormal equations; cf. S. Kovalevsky's example of the heat equation $\partial u/\partial t=\partial^2 u/\partial x^2$, with $u(0, x)$ as given initial Cauchy data. [See, e.g., S. Kovalevskaya, Scientific works, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1948; these Rev. **14**, 121.] The inverse Cauchy problem consists, essentially, in the determination of necessary and sufficient conditions on the Cauchy data (2) in order that the corresponding solution of (1) be analytic in the "principal" variable t. Section 1 contains a formulation of the problem and section 2 deals with infinitely differentiable functions, in particular, those of Gevrey's [Ann. Sci. Ecole Norm. Sup. (3) **35**, 129–190 (1918)] classes α. Section 3 discusses "two term" equations, e.g., $\partial^p u/\partial t^p=\partial^q u/\partial x^q$, and the concept of "weight" of an equation. Section 4 deals with the method of majorants and also with successive approximations [cf. Fridlender, Doklady Akad. Nauk SSSR (N.S.) **76**, 363–365 (1951); these Rev. **14**, 284]. Section 5 considers certain existence theorems for integro-differential equations [cf. Kokareva, ibid. **79**, 13–16 (1951); these Rev. **14**, 284]. Section 6 is concerned with the inverse Cauchy-Kovalevsky problem for systems of linear equations. The remaining four sections are devoted to related questions, and to the formulation of unsolved problems. *J. B. Dias* (College Park, Md.).

Hartman, Philip, and Wintner, Aurel. On the inverse of the parabolic differential operator $\partial^2/\partial x^2-\partial/\partial t$. Amer. J. Math. **75**, 598–610 (1953).

Let D be used to denote the interior of the rectangle R $(-1\leq x\leq 1, 0\leq y\leq 1)$. With $\partial_h u$ defined by

$$h^2\partial_h u=\{u(x+h, y)-2u(x, y)+u(x-h, y)\}$$
$$-\{u(x, y)-u(x, y-h^2)\}$$

the operator $\partial^* u = \lim \partial_\lambda u$ is a generalization of the operator $\partial u = u_{xx} - u_y$. The object of the present paper is to bring the local theory for the parabolic equations $\partial u = f(x, y)$ and $\partial^* u = f$ to the same stage of development as that for the corresponding elliptic equations. Four of the results obtained are listed here. (1) If u_1 and u_2 are two continuous solutions of $\partial^* u = f$, then $v = u_1 - u_2$ is of class C^∞ and v is a solution of $\partial v = 0$. (2) There exist on D continuous functions $f(x, y)$ corresponding to which $\partial u = f$ has no continuous solution $u(x, y)$ on any open subset of D. In contrast to this the following theorem is proved. (3) If f is continuous on R, then $\partial^* u = f$ possesses on D continuous solutions. (4) Assuming f continuous on R and that u is a continuous solution of $\partial^* u = f$ on D, the authors give necessary and sufficient conditions for the existence of the partial derivatives u_y and u_{xx}. *F. G. Dressel* (Durham, N. C.).

Karimov, D. H. On an equation of parabolic type. Doklady Akad. Nauk UzSSR **1949**, no. 4, 6–8 (1949). (Russian. Uzbek summary)

See C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 119–121 (1947); Doklady Akad. Nauk SSSR (N.S.) **58**, 969–972 (1947); these Rev. **9**, 37, 287.

Fenyő, István. Sur une méthode de solution de quelques équations différentielles de la physique mathématique. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 355–362 (1953). (Hungarian. Russian and French summaries)

Certain boundary-value and initial-value problems concerning partial differential equations (with two independent variables) can be solved by expanding the unknown function in terms of suitable orthogonal functions of the one variable. For the coefficients ordinary differential equations are obtained. This method is illustrated 1) by the diffusion problem

$$\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} = \frac{\partial y}{\partial t}, \quad y = y_0 \text{ for } r = 1,\ t > 0, \quad y = 0 \text{ for } r < 1,\ t = 0,$$

and 2) by the problem of distribution of electricity on a conducting circular disk. *G. Szegő* (Stanford, Calif.).

Manfredi, Bianca. Sopra un problema non lineare di propagazione del calore per un mezzo dotato di simmetria sferica. Rivista Mat. Univ. Parma **4**, 123–132 (1953).

The interior K of a sphere of radius $r = a$ is kept at constant temperature one. In the region S $(r > a)$ exterior to K the temperature is given by the function $U = U(r, t)$. The Laplace transform is used to reduce the following heat conduction problem to a nonlinear Volterra integral equation. In S the function U is to satisfy the conditions $U_{rr} + 2U_r/r = U_t$, $0 < U < 1$, $U(r, 0) = 0$, and on the boundary $r = a$ of S the (nonlinear) condition

$$-U_r = (1 - U)f(U) = G(U)$$

is to hold. $G(\xi)$ is a monotone decreasing continuous function of ξ with $G(1) = 0$. In the last section of the paper it is stated that this problem has a unique solution if $G(\xi)$ satisfies a Lipschitz condition on $0 \leqq \xi \leqq 1$. In an earlier paper the author treated a similar problem for the region exterior to a cylinder [same Rivista **3**, 383–396 (1952); these Rev. **14**, 1090; the following reference should be added to those listed in that review: J. H. Roberts, Proc. Amer. Math. Soc. **4**, 640–644 (1953); these Rev. **15**, 38]. *F. G. Dressel.*

Datzeff, Assène. Sur la propagation de la chaleur dans une barre non homogène. Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie II. **47**, 1–32 (1952). (Bulgarian summary)

Let a non-homogeneous bar of length 1 extend along the x-axis from $x = x_0$ to $x = x'$ and let the lateral surface of the bar be impermeable to heat. The temperature $u(x, t)$ is defined by the boundary value problem

$$(1) \qquad \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) = \rho\sigma\frac{\partial u}{\partial t},$$

where k, ρ and σ are functions of x only, and

$$(2) \qquad u(x, 0) = \Phi(x) \quad (x_0 < x < x'),$$

$$(3) \qquad u(x_0, t) = \phi(t), \quad u(x', t) = \psi(t) \quad (t > 0),$$

where Φ, ϕ and ψ are arbitrary bounded and integrable functions.

The method of solution is as follows: divide the interval (x_0, x') into n parts by the selection of abscissas $x_0, x_1, x_2, \cdots, x_n = x'$ and in the interval (x_{i-1}, x_i) $(i = 1, 2, \cdots, n)$ replace the functions $k(x)$, $\rho(x)$, $\sigma(x)$ by the constant values $k_{in} = k(x_{i-1})$, $\rho_{in} = \rho(x_{i-1})$, $\sigma_{in} = \sigma(x_{i-1})$, respectively. Then in the interval (x_{i-1}, x_i) equation (1) is replaced by

$$(4) \qquad a^2_{in}\frac{\partial^2 u_{in}}{\partial x^2} = \frac{\partial u_{in}}{\partial t}, \quad a^2_{in} = \frac{k_{in}}{\rho_{in}\sigma_{in}} \quad (i = 1, 2, \cdots, n),$$

where u_{in} is the temperature function in the interval (x_{i-1}, x_i). The conditions that must be satisfied at the interior points of the subdivision are

$$(5) \qquad u_{in}(x_i, t) = u_{i+1,n}(x_i, t),$$

$$(6) \qquad k_{in}\frac{\partial u_{in}(x_i, t)}{\partial x} = k_{i+1,n}\frac{\partial u_{i+1,n}}{\partial x}(x_i, t).$$

At the ends it is required that

$$(7) \qquad u_{1n}(x_0, t) = \phi(t), \quad u_{nn}(x', t) = \psi(t), \quad (t > 0).$$

The solutions to problems (4) subject to the appropriate conditions from (5), (6) and (7) are found by a method discussed in previous papers. [See Datzeff, these Rev. **9**, 146, 147; **12**, 263, 504].

The solution to the problem given by (1), (2) and (3) is found from the above by passing to the limit as $n \to \infty$. It is shown that: (a) the sequence $u_{in}(x, t)$ is bounded for fixed n, that is, $|U_{in}(x, t)| < M$ $(x_0 < x < x',\ t > 0;\ i = 1, 2, \cdots, n)$, (b) the $\lim_{n\to\infty} u_{in}(x, t)$ satisfies equation (1), (c) the sequence $\{u_{in}(x, t)\}$ has a limit as $n \to \infty$, (d) the boundary conditions are satisfied, and (e) the solution is unique. The particular case for which $\partial u/\partial t = 0$ is solved for the boundary conditions $u(x_0) = C_0$, $u(x') = C'$ and the solution is shown to be attainable as special case of the solution of the more general problem defined by (1), (2) and (3).

The results are extended to include the case for which the length of the bar is infinite. Another extension enables the author to apply the method to obtain a solution of the problem defined by (1), (2) with the boundary conditions (3) replaced by

$$(8) \qquad \begin{aligned} &\frac{\partial u}{\partial x}(x_0, t) + \alpha_1(t)u(x_0, t) + \beta_1(t) = 0, \\ &\frac{\partial u}{\partial x}(x', t) + \alpha_2(t)u(x', t) + \beta_2(t) = 0, \end{aligned}$$

where α_1, α_2, β_1, β_2 are bounded integrable functions for $t > 0$.

C. G. Maple (Ames, Iowa).

*Bergman, Stefan. **Multivalued solutions of linear partial differential equations.** Contributions to the theory of Riemann surfaces, pp. 229–245. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The well-known integral operator method for the investigation of solutions of linear partial differential equations, originated and developed by the author in numerous papers, is described in outline and applied to the study of multivalued solutions. *Z. Nehari* (St. Louis, Mo.).

Bergman, Stefan. On solutions of linear partial differential equations of mixed type. Amer. J. Math. **74**, 444–474 (1952).

This paper deals with elliptico-hyperbolic equations of the form (1) $\psi_{xx}+l(H)\psi_{yy}=0$; $l(H)=\sum_1^n a_n(-H)^n$, $a_1>0$, $a_2<0$; $Hl(H)<0$. In the elliptic half-plane one can reduce (1) to canonical form by introducing a new independent variable $\lambda=\lambda(H)$. The equation becomes: (2) $\psi_{\lambda\lambda}+\psi_{yy}+4N(\lambda)\psi_\lambda=0$. Here $N(\lambda)$ is defined for $\lambda<0$ and is singular for $\lambda=0$. On this function $N(\lambda)$ several function-theoretical restrictions are imposed; in particular, $N(\lambda)$ must be analytic for $-\infty<\lambda<0$, and admit an expansion of the form $N(\lambda)=(1/12)\lambda^{-1}+\beta_1\lambda^{-1/3}+\beta_2\lambda^{1/3}+\cdots$ for small negative λ. The "simplest case": $l(H)=-H$, $N(\lambda)=(12\lambda)^{-1}$ is given special attention.

In §§4–6 the author derives various integral operators transforming analytic functions of a complex variable into solutions of (2), and their inverse operators. The method is an adaptation of his well known treatment of linear elliptic equations with regular analytic coefficients.

The following version of the Cauchy problem is treated. Let $\chi_1(\theta)$, $\chi_2(\theta)$ be two functions defined in $0\leqq\theta\leqq\theta_0$ by converging power series. One looks for solutions ψ of (1) defined in a domain B in the elliptic half-plane which contains the segment I: $0<\theta<\theta_0$ on its boundary, and such that $\psi=\chi_1$, $\psi_\lambda=\chi_2$. The discussion of this problem in the small (B not prescribed) is elementary (§2). In §7 some results on the problem in the large are derived using the operators. The results are too complicated to be stated here. Connections with the theory of several complex variables are pointed out in §8. *L. Bers* (New York, N. Y.).

Bauer, Friedrich L. Gruppentheoretische Untersuchungen zur Theorie der Spinwellengleichungen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 111–179 (1953).

This is a systematic treatment of wave equations of the Dirac type by means of the theory of finite-dimensional representations of semi-simple Lie groups. Most of the material is expository, and in particular the paper contains a thorough account of relevant aspects of the theory of Lie algebras. *I. E. Segal* (New York, N. Y.).

Yosida, Kôsaku. On the existence of the resolvent kernel for elliptic differential operator in a compact Riemann space. Nagoya Math. J. **4**, 63–72 (1952).

This is essentially a continuation of the paper in same J. **3**, 1–4 (1951); these Rev. **13**, 560. Consider an elliptic differential operator Ω [in the notation of the cited review $\Omega=A'f+cf$] and assume that the Riemann manifold R and the coefficients b^{ij}, a^i, c are infinitely differentiable. The existence of a resolvent $(m-\Omega)^{-1}$ follows as in the paper cited. Now the author proves that, for m sufficiently large, this resolvent may be expressed as a regular transformation with a kernel $p_m(x, y)$. The latter can be interpreted as the Laplace transform of a transition probability, and thus the result has an immediate application to the diffusion process generated by Ω. In the special case of a symmetric operator the author obtains even a bilinear expansion for the transition probabilities, and proves certain regularity properties for them. *W. Feller* (Princeton, N. J.).

Alekseeva, O. P. A closed solution of the wave equation for a bounded medium. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 501–505 (1953). (Russian)

This paper gives a method of constructing closed solutions of boundary-value problems based upon the use of the Laplace-Carson transform. The problem of finding the potential in an electric line is given as an example. Explicitly, he solves the problem $\partial^2V/\partial x^2=LC\partial^2V/\partial t^2$ for the boundary conditions $V(0, t)=E$, $\partial V(h, t)/\partial x=0$, and the initial conditions $V(x, 0)=0$, $\partial V(x, 0)/\partial t=0$, and shows that it is possible to pass from this 'closed' form to the usual solution in terms of Fourier series. *C. G. Maple.*

Durand, Emile. Une identité conduisant à la solution du problème de Kirchhoff pour les ondes amorties. C. R. Acad. Sci. Paris **237**, 647–649 (1953).

The author proves the identity

$$A\psi(x_n, \nu t)=[\Delta+k_0^2-\nu^{-2}\partial_t\partial_t]\int\int\int_V d\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\times\int_{-\infty}^{t-r/\nu}\psi(x_n', \nu\tau)I_0(k_0\gamma)\nu d\tau$$

where $\gamma=[\nu^2(t-\tau)^2-r^2]^{1/2}$ and the factor A is 4π, 2π, or 0 when the point x_n is within, on, or outside the boundary, respectively, of the volume V. He shows how this identity can be used in the solution of the Kirchhoff problem for fields satisfying the equation

$$[\Delta+k_0^2-\nu^2\partial_t\partial_t]\psi=f(x_n, \nu t)$$

where f is given. *J. Shmoys* (New York, N. Y.).

Titchmarsh, E. C. Eigenfunction expansions associated with partial differential equations. II. Proc. London Math. Soc. (3) **3**, 80–98 (1953).

Titchmarsh, E. C. Eigenfunction expansions associated with partial differential equations. III. Proc. London Math. Soc. (3) **3**, 153–169 (1953).

[For part I see same J. (3) **1**, 1–27 (1951); these Rev. **13**, 241.] These papers are a continuation of the author's study of the eigenfunction expansion associated with the partial differential equation

$$(*)\qquad \nabla^2\phi+\{\lambda-q(x, y)\}\phi=0$$

when the region of consideration is the whole (x, y)-plane. In Parts II, III attention is restricted to $q(x, y)$ a continuous function that tends to infinity as $x^2+y^2\to\infty$, in which case the spectrum consists of discrete eigenvalues $\lambda_1, \lambda_2, \cdots$, where $\lambda_n\to\infty$ as $n\to\infty$.

In Part II the author establishes additional properties of the Green's function and results on the order of the eigenfunctions, as well as results on the convergence and summability of the eigenfunction series expansion.

Part III is devoted to a study of the asymptotic distribution of the eigenvalues of (*). If $N(\lambda)$ is the number of eigenvalues not exceeding λ, then by means of certain formulas involving the Green's function it is shown that

$$(**)\qquad \int_0^\mu\frac{N(\lambda)d\lambda}{(\lambda+\mu)^3}\sim\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dxdy}{q(x, y)+\mu}\quad\text{as}\quad\mu\to\infty$$

whenever $q(x, y)$ satisfies, with positive constants A, c, a, the following conditions:

(i) $0<q(x, y)<A \exp\{c(x^2+y^2)^{\frac{1}{2}}\}$;

(ii) $|q(x+r\cos\theta, y+r\sin\theta)-q(x, y)|<Ar[q(x, y)]^a$, $0<r\leqq 1$, $0<a<3/2$;

(iii) $q(x+r\cos\theta, y+r\sin\theta)<A \exp\{\frac{1}{2}r[q(x, y)]^{\frac{1}{2}}\}$.

Finally, if $q(x, y)$ satisfies the further condition that

$$\iint_{q>r}\frac{1}{q}dxdy=O\left(\frac{1}{r}\iint_{q\leqq r}dxdy\right).$$

it is shown by a Tauberian argument that (**) implies

$$(***)\qquad N(\lambda)\sim\frac{1}{4\pi}\iint_{q<\lambda}[\lambda-q(x, y)]dxdy.$$

W. T. Reid (Evanston, Ill.).

Vekua, I. N. Systems of differential equations of the first order of elliptic type and boundary value problems, with an application to the theory of shells. Mat. Sbornik N. S. 31(73), 217–314 (1952). (Russian)

This paper deals with systems of the form

$$(0)\qquad u_x-v_y=a_1u+b_1v+c_1,\quad u_y+v_x=a_2u+b_2v+c_2.$$

Set $u+iv=w$, $w_x+iw_y=2w_{\bar z}$. Then system (0) can be written in the complex form (0′) $w_{\bar z}=aw+b\bar w+c$. For the most part, the homogeneous equation

$$(1)\qquad w_{\bar z}=aw+b\bar w$$

is considered.

The (complex-valued) coefficients a, b, c are assumed only to be continuous. Hence there is no reason to expect continuously differentiable solutions. Rather, solutions "of class $C_{\bar z}$" are considered. A function $w(z)$ is called of class $C_{\bar z}$ if the Pompeiu derivative

$$w_{\bar z}(z_0)=\lim_{L\to z_0}\left\{[2i(\text{area of } T)]^{-1}\int_L w dz\right\}$$

(where T is the domain interior to a rectifiable Jordan curve L) exists and is continuous. In general, formal derivations based on the definition $w_{\bar z}=\frac{1}{2}(w_x+iw_y)$ can be justified for functions of class $C_{\bar z}$. The definition and properties of this clan are contained in §2. In particular the Dini-continuity of $C_{\bar z}$ functions is proved.

In §3 properties of solutions $w(z)$ of (1) are treated. If $w\not\equiv 0$, the zeros of w are isolated, in view of Carleman's classical theorem [C. R. Acad. Sci. Paris 197, 471–474 (1933)]. Using this the author finds for each w the representation

$$(2)\qquad w(z)=e^{\omega(z)}f(z),\ \omega \text{ continuous, } f \text{ analytic},$$

where ω can be expressed explicitly in terms of w and the domain considered. (This is a special case of a result due to the reviewer; see below.) This representation implies several corollaries, in particular the argument-principle for w.

§4 deals with transformations of (0′) resulting from a substitution $\hat w=w_0w$ (w_0 fixed function), or from a conformal mapping $z\to f(z)$. Also several integral identities are obtained. Finally, it is shown how system (0) can be connected with a second order equation

$$(3)\qquad \Delta u+\alpha u_x+\beta u_y+\gamma u+\delta=0.$$

The Laplace operator Δ in (3) is interpreted in a certain generalized way, based on the Pompeiu derivative. This is necessary, since the coefficients of (3) need not be Hölder-continuous.

In §5 solutions of (0′) and (1) are studied in a bounded domain T whose boundary L consists of $(m+1)$ rectifiable Jordan curves. It is assumed that $a\equiv 0$, which involves no serious loss of generality. If w satisfies (0′) and is continuous on L, then

$$(4)\qquad w(z)+\frac{1}{\pi}\iint\frac{b(\zeta)\overline{w(\zeta)}}{\zeta-z}d\xi d\eta=G(z)=\Phi(z)-\frac{1}{\pi}\iint\frac{c(\zeta)d\xi d\eta}{\zeta-z}$$

where Φ is continuous on $T+L$ and analytic in T. Here and hereafter all double integrals are taken over T. (3) is considered as an integral equation for w and it is shown that the homogeneous equation ($\Phi\equiv c\equiv 0$) has only the trivial solution. It follows that (3) is always solvable and the solution is given in the form

$$(5)\qquad w(z)=\phi(z)+\iint\{\Gamma_1(z, \zeta)\phi(\zeta)+\Gamma_2(z, \zeta)\overline{\phi(\zeta)}-(1/\pi)\Omega_1(z, \zeta)c(\zeta)-(1/\pi)\Omega_2(z, \zeta)\overline{c(\zeta)}\}d\xi d\eta$$

where $\Gamma_1=\sum_{1,\infty}K_{sj}$, $\Gamma_2=\sum_{1,s}K_{sj+1}$,

$$K_1(z, \zeta)=-b(\zeta)/[\pi(\zeta-z)],$$

$$K_{n+1}(z, \zeta)=\iint K_1(z, \zeta')\overline{K_n(\zeta', \zeta)}d\xi' d\eta',$$

$$\Omega_1(z, \zeta)=\frac{1}{\zeta-z}+\iint\frac{\Gamma_1(z, \zeta')}{\zeta-\zeta'}d\xi' d\eta',$$

$$\Omega_2(z, \zeta)=\iint\frac{\Gamma_2(z, \zeta')}{\zeta-\zeta'}d\xi' d\eta'$$

A detailed investigation of the kernels Γ, follows.

In §6 the representation (5) is used to derive the formula

$$(6)\qquad w(z)=\frac{1}{2\pi i}\int_L\Omega_1(z, \zeta)w(\zeta)d\zeta+\Omega_2(z, \zeta)\overline{w(\zeta)}\overline{d\zeta}$$

for a solution $w(z)$, $z\in T$, of (1) with $a\equiv 0$ ("Cauchy formula"). A study is made of the boundary values of w, and of integrals of the form (5) in which $w(\zeta)$ is replaced by an arbitrarily given Hölder-continuous function $\phi(\zeta)$ ("generalized Cauchy integrals").

In §7 formula (5) is used to obtain in T a complete sequence of particular solutions of (1). If T is a disc (punctured disc) this leads to expansions similar to Taylor series (Laurent series). This section also deals with compactness properties of solutions.

In §8 the previous results are used to treat a boundary-value problem (Problem A) for equation (1), with $a\equiv 0$, in the domain T. The boundary condition reads: $\alpha u+\beta v=\gamma$ on L. Here the given functions α, β, γ are Hölder-continuous on L, $\alpha^2+\beta^2\equiv 1$, and L consists of $(m+1)$ Lyapounov curves. For $\gamma\equiv 0$ we have the homogeneous problem (Problem A°). Let $2\pi n_j$ denote the increment experienced by $\arg(\alpha+i\beta)$ when the jth curve of L is traversed once in the direction positive with respect to T, and set $n=n_0+\cdots+n_m$. The main results are as follows: (1) If $n<0$, A° has no solution and A has a (unique) solution if $(-2n+m-1)$ linear conditions are satisfied. (2) If $n+1>m$, then A is always solvable, and A° has exactly $2n-m+1$ solutions. The author's

method consists in reducing the problem to a singular integral equation involving "generalized Cauchy integrals", and using the theory of such equations. For the Cauchy-Riemann equations such a treatment was given by Kveselava [Soobščeniya Akad. Nauk Gruzin. SSR 6, 581–590 (1945); these Rev. 8, 326]. Other topics in the § are: the "oblique derivative" problem for equation (2) with $\gamma \equiv \delta \equiv 0$, construction of a boundary value problem adjoint to A, independent treatment of the case $n \geqq 0$, $m = 0$.

Applications to the theory of elastic shells are given in §9. In §10, finally the case of coefficients a, b which are analytic in (x, y) is treated in the spirit of the author's book [New methods for solving elliptic equations, OGIZ, Moscow-Leningrad, 1948; these Rev. 11, 598].

[In §§2–7 of the paper there is a certain overlap with the reviewer's theory of pseudo-analytic functions [Proc. Nat. Acad. Sci. U. S. A. 36, 130–136 (1950); 37, 42–47 (1951); Theory of pseudo-analytic functions, New York University, 1953; these Rev. 12, 173; 13, 352; 15, 211]. In particular, representation (2) is a special case of one half of the so-called "similarity principle".] *L. Bers* (New York, N. Y.).

Fichera, Gaetano. Condizioni perché sia compatibile il problema principale della statica elastica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 14, 397–400 (1953).

Soit A un domaine borné de R^3, limité par une surface régulière S; soient S_1 et S_2 deux portions de S, limitées par une même courbe régulière. Le problème en question consiste à trouver un vecteur $\mathbf{u}$ satisfaisant à

$$\Delta \mathbf{u} + k \operatorname{grad} \operatorname{div} \mathbf{u} = \mathbf{f} \quad \text{dans } A$$

et aux conditions à la frontière: $\mathbf{u} = 0$ sur S, $L(\mathbf{u}) = \mathbf{g}$ sur S_2, avec

$$L(\mathbf{u}) = (k-1)(\operatorname{div} \mathbf{u})\nu + 2d\mathbf{u}/d\nu + \nu \wedge \operatorname{rot} \mathbf{u}$$

(ν désigne la normale à S dirigée vers l'intérieur, et k une constante $> 1/3$). L'auteur énonce sans démonstrations (celles-ci devant paraître ultérieurement) diverses conditions nécessaires de possibilité, d'où il résulte en particulier que les données $\mathbf{f}$ et $\mathbf{g}$ ne peuvent être arbitraires, mais doivent satisfaire à une relation linéaire de la forme $\int \mathbf{f} d\alpha + \int \mathbf{g} d\beta = 0$, où α et β sont des mesures vectorielles portées respectivement par les adhérences de A et de S_2. *J. Deny* (Strasbourg).

***John, Fritz. On behavior of solutions of partial differential equations.** The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1953. i+24 pp. (mimeographed)

The first part of the paper deals with Cauchy's problem for the hyperbolic equation $Lu = u_{tt} - a^2 u_{xx} - b^2 u_{yy} - ku = 0$ ($b \neq 0$) with data $u = f$ and $u_y = g$ in a bounded region R in the time-like plane $y = 0$. It is required that f and g be of class C^∞ in R and that u be of class C^∞ in the intersection of some open set containing R with the half-space $y \geqq 0$. If f and g are data for some u, write (1) $(f, g) \equiv 0$ (L). It is known that f and g determine u uniquely and that they can be given independently of each other provided that they are analytic. The validity of (1) for a given pair (f, g) depends strongly on the coefficients of L, and u has in general no continuation to negative y. If it has, however, and this is equivalent to $(f, 0) \equiv 0$ (L) and $(0, g) \equiv 0$ (L), then the same congruences are true when L is replaced by a similarly defined L' and $a^2 \geqq a'^2$. To a given f one can always by means of a Fourier transform find a $\bar{g}$ such that $(f, \bar{g}) \equiv 0$ (L). Hence in order to decide whether (1) is true for a given pair, it suffices to consider the case $f = 0$. Then the congruence does not depend on b, c, and k and by a linear transformation we may assume that L has the form $Lu = u_{tt} - u_{xx} - u_{yy}$. It is proved that a necessary and sufficient condition that $(0, g) \equiv 0$ (L) is that g be locally of the form

$$\int_0^{2\pi} G(x \cos\theta + t, \theta) d\theta,$$

where G and all its derivatives with respect to the first variable are continuous. When it exists, a solution u with Cauchy data 0, g is given by $u(x, y, t) = \int_{-\infty}^{+\infty} I(x, y, t, \lambda) d\lambda$, where

$$I = (2\pi)^{-1} y \int_{-1}^{+1} g(x - iyF(\eta, \lambda), t + \eta y)(2F/\lambda)^{-1/2} F_\lambda d\eta,$$

$(F = \lambda(1-\eta^2)/(1+(1+\lambda^2-\lambda^2\eta^2)^{\frac{1}{2}}))$. For small imaginary λ, I is well defined by g and can be shown to be analytic in λ for real λ. The second part of the paper deals with applications of the method of spherical means [F. John, Comm. Pure Appl. Math. 2, 209–253 (1949); these Rev. 12, 185]. The following rather general result is obtained. Let $u^1, \cdots$ be of class C^3 in a neighborhood of the origin and satisfy a system of differential equations with coefficients in C^∞ and assume that the system is not characteristic in the direction of one of the variables, say t. Then the expression $i\int_{-1}^{+1} (\beta^2 - \tau^2)^{m-1} u^k(\tau, .) d\tau$ can be differentiated m times with respect to the other variables. *L. Gårding* (Lund).

Grib, A. A. Generalization of the Euler-Darboux equation with integral coefficients. Doklady Akad. Nauk SSSR (N.S.) 90, 953–956 (1953). (Russian)

The author considers hyperbolic equations of the form

$$\frac{\partial^2 Z}{\partial \xi \partial \eta} + P(\xi, \eta)\frac{\partial Z}{\partial \xi} + Q(\xi, \eta)\frac{\partial Z}{\partial \eta} = R(\xi, \eta) Z.$$

If P, Q, R can be approximated by two functions $f_1(\xi)$, $f_2(\eta)$ and three constants A, B, C so that

$$P \sim A\frac{\partial}{\partial \eta} \ln t, \quad Q \sim B\frac{\partial}{\partial \xi} \ln t, \quad R \sim C\frac{\partial}{\partial \xi}\ln t \frac{\partial}{\partial \eta} \ln t,$$

where $t = f_1(\xi) + f_2(\eta)$, then solutions to the above equation may be approximated by solutions of the Euler-Darboux equation. Conditions are given on P, Q, R that they should be approximable in the above manner. By the same technique solutions to certain special non-linear hyperbolic equations are obtained. *M. H. Protter.*

Integral Equations

Platone, Giulio. Equazioni integrali a nuclei sommabili. Pont. Acad. Sci. Acta 15, 77–92 (1953). (Latin summary)

The integral equations discussed are:

$$\varphi(x) = \lambda \int_E A(y) k(x, y) \varphi(y) dy + f(x)$$

and its adjoint $\psi(x) = \mu A(x) \int_E \psi(y) k(y, x) dy + g(x)$, in which E is a measurable subset of r-dimensional euclidean space, $f(x)$ is essentially bounded on E, $k(x, y)$ is essentially bounded on $E \times E$, and $g(x)$ and $A(x)$ are Lebesgue integrable on E. It is shown that the iterated series converge

for $|\lambda|<1/LM$, where $\int_E|A(x)|dx=M$ and L is the almost upper bound of $|k(x,y)|$ on $E\times E$. Also if there exist continuous functions $\alpha_{in}(x)$, $\beta_{in}(y)$ such that the almost bound of $|\sum_{i=1}^{n}\alpha_{in}(x)\beta_{in}(y)-k(x,y)|$ approaches zero in n, then the usual nondeterminantal theorems of Fredholm apply. Obviously these results are special cases of equations involving linear bounded transformations on a linear normed space to itself [see, for instance, a paper by reviewer in Bull. Amer. Math. Soc. **37**, 185–212 (1931), pp. 194–196.] *T. H. Hildebrandt* (Ann Arbor, Mich.).

Chang, Shih-Hsun. A generalization of a theorem of Goursat and Heywood. Acad. Sinica Science Record **5**, 11–16 (1952). (Chinese summary)

The theorem that if two integral equation kernels $P(x,y)$ and $Q(x,y)$ are orthogonal then $D_{P+Q}(\lambda)=D_P(\lambda)D_Q(\lambda)$, where $D_P(\lambda)$ is the Fredholm determinant of P, is extended to the case where $k(x,y)=\text{l.i.m.}_{n=\infty}\sum_{i=1}^{n}k_n(x,y)$. Here k_n are in L^2, mutually orthogonal and such that $\sum_n\|k_n\|^2<\infty$ with $\|k_n\|^2=\int\int|k(x,y)|^2dxdy$, and the modified Fredholm determinant $D_k^*(\lambda)$, in which the diagonals of the defining determinants are zero, replaces $D_k(\lambda)$.

T. H. Hildebrandt (Ann Arbor, Mich.).

***Pogorzelski, Witold. Equations intégrales singulières.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 561–564. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

In connection with a problem of tides H. Poincaré arrived at an integral equation (in appearance, of the second kind), whose kernel $N(s,\sigma)$ has a polar singularity for $s=\sigma$ (here s, σ are lengths of arcs along the regular boundary C of a domain in the plane). Poincaré solved (1) by means of a certain transformation (P) (now well known) of the iterated integral, involving $N(s,\sigma)$ and its residue. The author has previously given a sufficient condition that a kernel N be closed. Without proofs and without statement of precise hypotheses, he gives indications of how the notion of closed kernels and the transformation (P) enable one to reduce the system

$$\int_0^s N_\nu(x,y)F_\nu[x,y,\phi_1(y),\cdots,\phi_n(y)]dy=f_\nu(x) \qquad (\nu=1,\cdots,n)$$

and the single equation

$$\int_0^s N(x,y)F[x,y,\phi(y),\phi'(y),\cdots,\phi^{(n)}(y)]dy=f$$

to another system and to another equation, each solvable by successive approximations. [See also Ann. Soc. Polon. Math. **24**, 75–87 (1952); these Rev. **14**, 181.]

W. J. Trjitzinsky (Urbana, Ill.).

Leonov, M. Ya. Solution of an integral equation of the theory of the Newtonian potential. Ukrain Mat. Žurnal **5**, 50–57 (1953). (Russian)

The author gives an explicit solution of the integral equation

$$(1)\quad u(x,y)=\int_S\int\rho(\xi,\eta)r^{-1}d\xi d\eta \qquad (r^2=(x-\xi)^2+(y-\eta)^2,\ S=\{x^2+y^2\leqq a^2\})$$

arising in the field of elasticity, u being assigned on S. One has on S: $u(x,y)=U(x,y,0)$, where $U(x,y,z)$ is potential of masses distributed on the plane region S. The resolution of (1) is carried out without the aid of the solution of a corresponding problem of Dirichlet, on the basis of some of the fundamental properties of the Newtonian potential.

W. J. Trjitzinsky (Urbana, Ill.).

Germay, R. H. Sur l'intégration, par la méthode des approximations successives, des systèmes normaux d'équations intégro-différentielles récurrentes. Bull. Soc. Roy. Sci. Liège **22**, 131–138 (1953).

This note is an extension of previous work of the author [same Bull. **18**, 250–258 (1949); Ann. Soc. Sci. Bruxelles. Sér. I. **66**, 125–130 (1952); **67**, 13–18 (1953); these Rev. **11**, 368; **14**, 765] to systems of recurrent integro-differential equations. The subject was first treated by L. Bruwier [Mém. Soc. Roy. Sci. Liège (3) **17**, no. 22 (1932), p. 43].

I. A. Barnett (Cincinnati, Ohio).

Germay, R. H. Sur les intégrales infiniment voisines des systèmes d'équations intégro-différentielles récurrentes de forme normale. Bull. Soc. Roy. Sci. Liège **22**, 198–210 (1953).

The purpose of this note is to extend to systems of recurrent integro-differential equations the results that were proved by the author for single equations of the same form [same Bull. **22**, 64–76 (1953), pp. 69–70; these Rev. **14**, 988]. *I. A. Barnett* (Cincinnati, Ohio).

Šerman, D. I. On properties of infinite systems of equations in problems of torsion of certain doubly connected profiles. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 470–476 (1953). (Russian)

The Saint-Venant torsion problems for a circular cylinder weakened by a symmetrically located longitudinal circular cavity or by two longitudinal circular cavities have been reduced by the author [Doklady Akad. Nauk SSSR (N.S.) **63**, 499–502 (1948); these Rev. **10**, 651] and by R. D. Stepanov and Šerman [Akad. Nauk SSSR. Inženernyĭ Sbornik **11**, 127–150 (1952); these Rev. **14**, 430] to the solution of certain Fredholm integral equations. The solution of the integral equations was made to depend on the solution of two systems of linear algebraic equations in infinitely many unknowns. This note gives a demonstration that these systems are completely regular and hence admit of an estimate of the error resulting from truncating the systems.

I. S. Sokolnikoff (Los Angeles, Calif.).

Functional Analysis

Takenouchi, Osamu. Sur les espaces linéaires localement convexes. Math. J. Okayama Univ. **2**, 57–84 (1952)

In this paper the author generalizes earlier work of J. Dieudonné, L. Schwartz and G. Köthe on countable inductive limits of locally convex vector spaces and studies the general notions of inductive and projective limits of these spaces. He establishes some duality properties connecting these two notions and discusses the possibility of representing a locally convex vector space and its dual as projective or inductive limits of Banach spaces, with special emphasis on the spaces which are "bornologique" (=relatively strong and boundedly closed) in the sense of Mackey and "tonnelé" in the sense of Bourbaki. Some of the notions and results in this article have also appeared independently in work of W. F. Donoghue Jr. and K. T. Smith [Trans. Amer. Math.

Soc. 73, 321–344 (1952); these Rev. 14, 182] and in N. Bourbaki's "Espaces vectoriels topologiques", Chap. I and II [Actualités Sci. Ind., no. 1189, Hermann, Paris, 1953; these Rev. 14, 880]. The author states without proof that, in case the $f_{\alpha\beta}$ in the definition of inductive limits are homeomorphisms into, each f_γ is a homeomorphism into too. Bourbaki [loc. cit.], however, states this as an open question, as the proof given by Dieudonné and Schwartz in the countable case is not known to work otherwise.

L. Nachbin (Los Angeles, Calif.).

Amemiya, Ichiro. Quelques généralisations des théorèmes concernant aux produits directs des espaces linéaires localement convexes. Math. J. Okayama Univ. 2, 185–189 (1953).

Improving a result of Takenouchi [cf. the preceding review], the author shows that the cartesian product of a regular cardinal number of "bornologique" spaces is again "bornologique". As the author himself acknowledges, the same fact has been obtained by Donoghue and Smith [cited in the above review] who, completing an earlier result of Mackey, have shown more generally that the cartesian product $\prod_{i\in I}E_i$ of "bornologique" spaces E_i, $\dim E_i \geqq 1$ is "bornologique" if and only if the Ulam measure problem on I has no solution. *L. Nachbin* (Los Angeles, Calif.).

Schwartz, Laurent. Homomorphismes et applications complètement continues. C. R. Acad. Sci. Paris 236, 2472–2473 (1953).

The Riesz theory for $w=\lambda x+v$, v completely continuous, has been generalized to a locally convex linear topological space (l.t.s.), E, by Leray [Acta Sci. Math. Szeged 12, Pars B, 177–186 (1950); these Rev. 12, 32]. The present paper may be described as extending some of the results by (a) admitting a different range space F (again a convex l.t.s.) and replacing λx by u, a linear isomorphism into $u(E)$ with $u(E)$ closed. *D. G. Bourgin* (Urbana, Ill.).

Allen, H. S. Duality of the spaces of linear functionals on dual vector spaces. Bull. Soc. Math. France 80, 233–235 (1952).

Ruston, A. F. Note on a paper by H. S. Allen. Bull. Soc. Math. France 81, 77 (1953).

Let E and F be left and right linear vector spaces over a sfield K and let (x, y) be a non-degenerate bilinear functional defined on $E\times F$ to K. Then E and F are said to be "dual" spaces relative to (x, y) [Dieudonné, same Bull. Soc. 70, 46–75 (1942); these Rev. 6, 144; Jacobson, Ann. of Math. (2) 48, 8–21 (1947); these Rev. 8, 433]. Let F^* and E^* be the left and right K-spaces whose elements are linear functionals on F and E, respectively. The author proves that if K has characteristic different from 2, then F^* and E^* are dual spaces. The restriction on the characteristic of K in this result is removed by A. F. Ruston in the second paper. A left (or right) vector space E over K is said to be "self-dual" if there is an involution $a\to a^J$ in K and a scalar product (x, y) defined on $E\times E$ to K such that $(y, x)=\varepsilon(x, y)^J$ where $\varepsilon=\pm 1$ independently of x, y. A self-dual space is called "symplectic" if $(x, x)=0$ for all x and is called "unitary" if $(x, x)\neq 0$ for some x in E [Rickart, Amer. J. Math. 73, 697–716 (1951); Bull. Amer. Math. Soc. 57, 435–448 (1951); these Rev. 13, 532]. If E is unitary, there is no loss in taking $\varepsilon=1$. The author then proves that if K is not of characteristic 2 and if E is self-dual, then E^* is also self-dual and is symplectic or unitary along with E.

C. E. Rickart (New Haven, Conn.).

Audin, Maurice. Sur les transformations linéaires dans les espaces de Banach et l'alternative de Fredholm. C. R. Acad. Sci. Paris 237, 511–512 (1953).

The author studies the inequalities

$$1/\|A\| \leqq r_A \leqq \rho_A \leqq R_A \leqq \infty,$$

where A is a linear transformation on a Banach space and r_A, ρ_A, R_A are the upper bounds of $\|A-L\|^{-1}$, $\lim_{n\to\infty}\|(A-L)^n\|^{-1/n}$, $\lim_{n\to\infty}\|(A-V)^n\|^{-1/n}$ for finite-dimensional L and completely continuous V. He terms ρ_A the "Schmidt-radius", while r_A, R_A are two concepts of the "Fredholm-radius". He appears to conjecture (in "problem (II)") that $\rho_A=R_A$, which in the reviewer's opinion is certainly provable by methods of spectral theory if A is bounded. *F. V. Atkinson* (Ibadan).

Gohberg, I. C. On the index of an unbounded operator. Mat. Sbornik N.S. 33(75), 193–198 (1953). (Russian)

The author continues work of Kreĭn and Krasnosel'skiĭ [Mat. Sbornik N.S. 30(72), 219–224 (1952); these Rev. 13, 849] who made extensions of certain results of the reviewer [ibid. N.S. 28(70), 3–14 (1951); these Rev. 13, 46] concerning the "index" $\varkappa(A)$ of a linear operator. The present extension is of the law $\varkappa(AB)=\varkappa(A)+\varkappa(B)$, without now proving that AB is normally soluble. This enables him to improve on Kreĭn's and Krasnosel'skiĭ's extension of a stability property; the same has been done by a different method by B. Sz.-Nagy [Acta Math. Acad. Sci. Hungar. 3, 49–52 (1952); these Rev. 14, 564]. *F. V. Atkinson.*

Aržanyh, I. S. Parametric representation of solutions of a system of linear functional equations in commutative operators. Uspehi Matem. Nauk (N.S.) 8, no. 3(55), 157–160 (1953). (Russian)

The author considers equations of the form

$$\sum_{\mu=1}^{n} A_{\nu\mu}U_\mu = 0 \quad (\nu=1, \cdots, n),$$

where the $A_{\nu\mu}$ are linear commutative operations (operators which are elements of a commutative ring). He observes that, if $\Phi_1, \cdots, \Phi_n$ are solutions of the equation $\Delta\Phi=0$, where $\Delta=\text{determinant } A_{\nu\mu}$, and if $(-1)^{\nu+\mu}\Delta_{\nu\mu}$ is the cofactor of $A_{\nu\mu}$ in the matrix $\|A_{\nu\mu}\|$, then the above equations are satisfied by

$$U_\nu = \sum_{\mu=1}^{n}(-1)^{\nu+\mu}\Delta_{\nu\mu}\Phi_\mu \quad (\nu=1, \cdots, n).$$

To illustrate the applications of this result, the author uses it to obtain general solutions to the Cauchy-Riemann equations, Oseen's equations, and the equations of linear elasticity. *J. L. Ericksen* (Washington, D. C.).

Rosenbloom, Paul C. Quelques classes de problèmes extrémaux. Bull. Soc. Math. France 79, 1–58 (1951); 80, 183–215 (1952).

These papers contribute to "hard" analysis by methods of "soft" analysis (abstract spaces, convex sets) in the Hardy terminology. The guiding ideas are intuitively obvious and are given explicit recognition under the names "Principes du sommets, de l'arête et de déformation". For a convex polyhedron K they are: (s) for f continuous, $f(x)$ assumes its maximum at an extreme point (sommet) e of K; (a) a plane of support, not parallel to any face (l'arête) of K touches K in a unique extreme point; (d) under continuous deformation of faces, $\{A(t)\}$, and support plane, $S(t)$, with $S(t)$ never parallel to any $A(t)$, $e(t)$ of (a) varies continuously,

i.e., $e(t)$ remains the "same" extreme point in the obvious sense. These three principles have counterparts in infinite-dimensional spaces, say Banach spaces under compactness conditions, say w or w^* compactness, and the author enunciates a number of theorems under conditions guaranteeing the convex sets are regularly convex. Define F as the class of functions f representable in the form $f(t)=\sum a_n\phi_n(t)$, $a_n\geqq 0$ where $\phi_n \in C(0, 1]$ with $\lim_{n\to\infty}\phi_n(x)=0$, $x \in (0, 1)$ and $\phi_n(1)=1$. Then F is a convex subset of l. The system is completely Tchebysheffian, T, if (a) is valid and this is equivalent to a condition on the zeros of finite linear combinations of $\{\phi_n(t)\}$. If $a_n\neq 0$ for at most a finite set of n values, f is a ϕ nome. $\{\phi_n\}$ is a system of Descartes, D, if the variation in sign of a ϕ nome is an upper bound for the number of roots in $(0, 1]$. The terminology is the natural one for the special case $\phi_n(t)=t^n$ central for absolutely monotone functions. Let $0<x_1<x_2<\cdots<x_k\leqq 1$ and introduce the partition $\{i\,|\,1\leqq i\leqq k\}=E+L+U$, where (4) $f(x_i)=C_i$, $\leqq C_i$, $\geqq C_i$ according as $i \in E$, L or U. When $\{\phi_n\}$ is T, there is a ϕ nome, P, satisfying $P_k(1)\leqq f(1)$ for all $f \in F$, subject to (4), and if $x_k<1$, P is called the Principal Function of F. Application to the case $f(x)$ is a mean module over a circle of radius x for P_2 yields an inequality for the quadratic mean of an analytic function. P_2 leads to a four circle theorem, etc.

A charge μ is a finite additive set function on a Borel field and determines an element F of E^* for E the Banach space of real bounded functions $x=x(s)$ on Y with the usual norm. Let

$$M=\{\mu\,|\,F(x_i)=C_i,\ 1\leqq i\leqq m;\ F(x_i)\leqq C_i,\ m+1\leqq i\leqq k;\ \mu\geqq 0\}.$$

Then M is regularly convex. A central problem is the determination of an element of M for which $F(x_{k+1})$ is maximum. If μ takes on just two values, normalized to 0 and 1, it is an elementary charge (which is atomic under compactness of the underlying space) and an extreme point is a linear combination of at most k elementary charges with nonnegative coefficients. The heavily overworked terms regular and completely regular here characterize a clear notion of general position of $\{x_i\}$ under which there is a unique maximizing μ. If $\{x_i\,|\,1\leqq i\leqq k+1\}$ is T then (a) is valid and this implies Y is essentially linear. One type of specialization is to functions representable as $f(t)=\int K(t, s)d\mu(s)$ where K is given (and is for instance a Laplace kernel, Poisson kernel, etc.) and μ is nonnegative. Applications here are to completely monotone functions and to moduli of analytic functions in rings, etc. *D. G. Bourgin* (Urbana, Ill.).

Cronin, Jane. Analytic functional mappings. Ann. of Math. (2) **58**, 175–181 (1953).

The essence of this paper is the theorem that if $z_j'=\sum_{k(j)}^{\infty}P_{j,m}(z_1, \cdots, z_n)$, $j=1, \cdots, n$, where $P_{j,m}$ is a homogeneous mth degree polynomial in the n complex variables $\{Z_i\}$ then the topological degree of this mapping at 0, assuming $P_{j,k(j)}\neq 0$ for each j, is $\geqq\prod_{j=1}^{n}k(j)$. The equality holds if the resultant of $P_{j,k(j)}$ is not 0. *D. G. Bourgin.*

Redheffer, R. M. Operators and initial-value problems. Proc. Amer. Math. Soc. **4**, 617–629 (1953).

Let ϕ_{ab}, $b>a$, denote a two-parameter family of linear operators defined on a subset of the totality of complex-valued functions $f(x)$ on $(-\infty, \infty)$. It is said to be bounded pointwise by M_{ab} if $\phi_{ab}f$ exists for $f \in L^2$ and $|\phi_{ab}f|\leqq M_{ab}\int|f|^2$, bounded in mean by N_{ab} if $\phi_{ab}f$ is measurable and $\int|\phi_{ab}f|^2\leqq N_{ab}(\int|f|^2)$ for $f \in L^1\cap L^2$. The author proves the following four theorems. (1) The operators satisfying the above 3 conditions are characterized by the properties: (i) $\phi_{ab}f(y)=\int\lambda_{ab}(x, y)f(x)dx$; (ii) $\int|\lambda_{ab}(x, y)|^2dx\leqq M_{ab}$; (iii) $\int|\lambda_{ab}(x, y)|^2dy\leqq N_{ab}$ for almost all x;

$$\text{(iv)}\qquad \int\lambda_{ab}(x, y)\lambda_{bc}(y, z)dy=\lambda_{ac}(x, z)$$

for almost all x. (2) Let ϕ_{ab} be bounded pointwise by M_{ab}, closed, non-vanishing, spatially homogeneous, i.e.,

$$\phi_{ab}f(x-c)=g(y-c) \quad \text{if} \quad \phi_{ab}f(x)=g(y),$$

and let $\phi_{ab}f$ be measurable in a. Such ϕ_{ab} is characterized by the properties: (i) $\phi_{ab}=T^{-1}\Lambda_{ab}T$, where T denotes the Fourier transform in the L^2-sense;

$$\text{(ii)}\qquad \Lambda_{ab}(u)=\exp\left(\int_a^b d_x\alpha(x, u)\right)$$

for some function α of bounded variation in x on every finite interval; (iii) $\int|\Lambda_{ab}(u)|^2du\leqq M_{ab}$. (3) Let ϕ_{ab} be bounded pointwise by M_{ab}, closed, temporally homogeneous ($\phi_{ab}=\phi_{a+c,b+c}$), and let $\phi_{ac}f$ be continuous in a. Such ϕ_{ab} is characterized by the properties: (i) $\phi_{ab}=T^{-1}\Lambda^{b-a}T$; (ii) $\int|\Lambda(u)|^{2(b-a)}du\leqq M_{ab}$ for $b>a$. (4) Let

$$U(x, y)=T^{-1}F\Lambda^x=(2\pi)^{-1/2}\int e^{iux}\Lambda(t)^xF(t)dt,$$

where $\Lambda(t)$ is measurable, satisfy one and the same partial differential equation for all F giving sufficiently strong convergence of the integral. Then (i) $\log\Lambda(t)$ is equal almost everywhere to one and the same algebraic function of t, except when $\Lambda=0$ or when Λ is undefined; (ii) $U(x, y)$ satisfies one and the same linear homogeneous partial differential equation with constant coefficients. *K. Yosida.*

Gál, I. S. On sequences of operations in complete vector spaces. Amer. Math. Monthly **60**, 527–538 (1953).

In this expository article the author discusses the principles of uniform boundedness and of condensation of singularities for Banach spaces and their applications to analysis. *G. G. Lorentz* (Detroit, Mich.).

Beurling, Arne. A theorem on functions defined on a semigroup. Math. Scand. **1**, 127–130 (1953).

This paper is concerned with the question: When does the linear closed extension of the translates of a function defined on a semi-group S contain at least one character of S? Here S is the multiplicative semi-group of positive reals $\leqq 1$ and the function space is $L^p=L^p(0, 1)$. To each $f \in L^p$, $p>1$, is assigned the closed linear manifold C_f, $1\leqq r\leqq p$, spanned by the set $[f(x\xi);\ \xi \in S]$ in the topology of L^r. The "closure" of the translates is now taken to be $\Gamma_f{}^p=\bigcap[C_f{}^r\cap L^p;\ 1\leqq r<p]$. A continuous character of S belonging to L^p and suitably normalized has the form $x^{-\lambda}$ where $\mathrm{Re}(\lambda)<1/p$. The following theorem is proved: Let $f(x)$ belong to a space L^p, $1<p<\infty$, and let it not vanish almost everywhere on any interval $0<x\leqq a$. Then $\Gamma_f{}^p$ contains at least one function of the form $x^{-\lambda}$, $\mathrm{Re}(\lambda)<1/p$. *R. S. Phillips* (New Haven, Conn.).

Fantappiè, Luigi. Su un'espressione generale dei funzionali lineari mediante le funzioni "para-analitiche" di più variabili. Rend. Sem. Mat. Univ. Padova **22**, 1–10 (1953).

Soit A un compact de l'espace numérique complexe C^n (nous adoptons ici les notations de N. Bourbaki). La région linéaire (A) est constituée par les fonctions $f(z_1, \cdots, z_n)$ localement analytiques, dont les domaines $D(f)$ contiennent A.

Identifions chaque point $(z_1, \cdots, z_n)$ de $\mathbf{C}^n$ avec un point $(x_1, \cdots, x_{2n})$ de $\mathbf{R}^{2n}$, en posant $z_r = x_r + ix_{n+r}$ $(r=1, \cdots, n)$, et considérons la formule

$$(1) \qquad F(f) = \int_V f(z_1, \cdots, z_n)\omega, \quad f \in (A),$$

où V est une hypersurface de $\mathbf{R}^{2n}$, frontière d'un ouvert contenu dans $D(f)$ et

$$\omega = \sum_{r=1}^{n} (-1)^{r-1} u_r dx_1 \cdots dx_{r-1} dx_{r+1} \cdots dx_{2n},$$

les coefficients u_r étant des fonctions de $x_1, \cdots, x_{2n}$, définies et continûment différentiables dans un ouvert

$$\Omega \supset \Omega^* = D(f) \cap (\mathbf{R}^{2n} - A).$$

Soit d'autre part V' une varieté homologue à V contenue dans Ω^*. L'auteur démontre le résultat suivant: pour que $\int_V f\omega = \int_{V'} f\omega$, il faut et il suffit que

$$\sum_{r=1}^{n} \frac{\partial u_r}{\partial x_r} + i \sum_{r=1}^{n} \frac{\partial u_r}{\partial x_{n+r}} = 0, \quad u_{n+r} = iu_r \quad (r=1, 2, \cdots, n).$$

Ces conditions étant vérifiées, la formule (1) définit une fonctionnelle F linéaire et analytique dans (A), indépendante de V. On dit alors que le point $\mathbf{u} = (u_1, \cdots, u_n)$ de $\mathbf{C}^n$ est une fonction para-analytique de $\mathbf{z} = (z_1, \cdots, z_n)$; en particulier, pour $n=1$, on retrouve le cas des fonctions analytiques ordinaires.

L'auteur avance cette conjecture, que les formules du type (1) offrent un moyen de représenter toutes les fonctionnelles analytiques linéaires définies dans (A). Jusqu'ici on n'a réussi a trouver une telle représentation générale que dans le cas où A est un produit de sous-ensembles de $\mathbf{C}$ (au moyen de la formule intégrale de Fantappiè, déduite de celle de Cauchy).

L'ensemble $\mathfrak{P}$ des fonctions $\mathbf{u}$ para-analytiques définies dans Ω est un module sur l'anneau $\mathfrak{A}$ des fonctions $f(z_1, \cdots, z_n)$ analytiques dans Ω (en donnant à $\mathbf{u}+\mathbf{v}$ et $f\mathbf{u}$ le sens usuel). En outre, les fonctions $\mathbf{u} \in \mathfrak{P}$ qui sont indéfiniment dérivables dans Ω par rapport aux x_r forment un sous-module de $\mathfrak{P}$, où l'auteur définit encore un produit $(\mathbf{u}, \mathbf{v})$ qui le fait devenir une algèbre de Lie sur l'anneau $\mathfrak{A}$.

J. Sebastião e Silva (Lisbonne).

Varsano, Sami. Sui funzionali analitici lineari del ciclo chiuso delle funzioni di più variabili. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 123–135 (1950).

Ce travail généralise aux fonctions de plusieurs variables le concept de "opérateur du cycle fermé" étudié par Fantappiè dans le cas analytique pour les fonctions d'une seule variable. Soit $\mathfrak{S}^{(n)}$ l'espace des fonctions $\varphi(\mathbf{z}) = (z_1, \cdots, z_n)$ birégulières; on nomme opérateurs du cycle fermé les applications $F(\varphi) = \psi$ de parties de $\mathfrak{S}^{(n)}$ dans $\mathfrak{S}^{(n)}$ telles que: (α) $F_{\mathbf{z}}[\varphi(\mathbf{z}+\mathbf{h})] = \psi(\mathbf{z}+\mathbf{h})$ quel que soit $\mathbf{h} = (h_1, \cdots, h_n) \in \mathbf{C}^n$. Si F est un opérateur linéaire analytique défini dans une region (A), où A est un produit cartésian de compacts de $\mathbf{C}$, la condition (α) est équivalente à chacune des conditions suivantes: (β) $F[\partial\varphi/\partial z_s] = \partial\psi/\partial z_s$, $s=1, \cdots, n$; (γ) l'indicatrice de F, donnée par $F_{\mathbf{z}}[(\lambda_1 - z_1)^{-1} \cdots (\lambda_n - z_n)^{-1}]$, est une fonction $u(\lambda_1 - z_1, \cdots, \lambda_n - z_n)$ des différences $\lambda_1 - z_1, \cdots, \lambda_n - z_n$. L'auteur considère plus généralement des opérateurs linéaires analytiques du type

$$F_{\mathbf{z}}[\varphi(z_1, \cdots, z_n)] = \psi(\zeta_1, \cdots, \zeta_m),$$

pour lesquels on peut faire une partition des variables $\zeta_1, \cdots, \zeta_m$ en $n+1$ classes

$$\{\zeta_{s1}, \cdots, \zeta_{sk_s}\}, \quad k_1 + \cdots + k_n + k_{n+1} = m,$$

de façon que: (α') chaque accroissement h_s de z_s dans φ détermine le même accroissement h_s de $\zeta_{s1}, \cdots, \zeta_{sk_s}$ dans ψ $(s=1, \cdots, n)$. L'auteur montre que (α') est équivalente à: (β') $F[\partial\varphi/\partial z_s] = \sum_{r=1}^{k_s} \partial\psi/\partial x_{sr}$, $s=1, \cdots, n$; et il trouve encore une condition équivalente à (α') analogue à (γ). Enfin il détermine la forme de l'indicatrice projective des opérateurs linéaires analytiques vérifiant (α).

J. Sebastião e Silva (Lisbonne).

Pellegrino, Franco. Sui funzionali del ciclo chiuso più generali. Collectanea Math. **2**, 87–113 (1949).

Une fonctionnelle $F_t[y(t), \lambda_1, \cdots, \lambda_r]$ de $y(t)$ et des variables complexes $\lambda_1, \cdots, \lambda_r$ est dite "du cycle fermé" dans un domaine $\mathfrak{D}$ de l'espace fonctionnel analytique $\mathfrak{S}$, si l'on a pour toute fonction $y \in \mathfrak{D}$:

$$(\alpha) \quad F_t[y(t+\omega), \lambda_1, \cdots, \lambda_r] = F_t[y(t), \lambda_1+\omega, \cdots, \lambda_r+\omega],$$

quels que soient $\omega, \lambda_1, \cdots, \lambda_r$. L'auteur établit le résultat suivant: Pour qu'une fonctionnelle analytique

$$F_t[y(t), z_1, \cdots, z_m]$$

soit du cycle fermé dans le domaine de convergence de son développement en série de Taylor généralisée, autour d'un point $y_0(t)$ donné, il faut et il suffit que cette fonctionnelle et toutes ses dérivées, fonctions de y et de $z_1, \cdots, z_m, \zeta_1, \cdots, \zeta_n$ données par l'expression

$$\frac{\partial}{\partial\varepsilon_1} \cdots \frac{\partial}{\partial\varepsilon_n} F_t\left[y(t) + \sum_{k=1}^{n} \frac{\varepsilon_k}{\zeta_k - t}; z_1, \cdots, z_m\right]\Bigg|_{\varepsilon_1 = \cdots = \varepsilon_n = 0},$$

vérifient la condition (α) pour le seul point y_0. Ensuite l'auteur étudie en détail le cas des fonctionnelles polynomiales analytiques, dont il détermine l'expression générale. La nouveauté essentielle de ces résultats consiste au fait qu'ils sont applicables à l'étude de fonctionnelles qui ne sont pas définies pour les constantes, comme par exemple les fonctionnelles F qui vérifient l'équation

$$F[y(t+h)] = F[y(t)] - h$$

et dont les dérivées sont des fonctionnelles du cycle fermé.

J. Sebastião e Silva (Lisbonne).

Pellegrino, Franco. Ancora sulla continuità dei funzionali analitici. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 104–122 (1950).

L'auteur se propose de perfectionner ici le contenu d'un travail précédent [Haefeli et Pellegrino, Comment. Math. Helv. **21**, 225–246 (1948); ces Rev. **9**, 515] où il a donné, en collaboration avec Haefeli, une démonstration "a priori" de la continuité des fonctionnelles analytiques linéaires, basée sur un critère général de continuité des fonctionnelles analytiques. Mais le reviewer avait déjà établi que toutes les fonctionnelles analytiques de Fantappiè sont continues [Thèse, 1948; Portugaliae Math. **9**, 1–130 (1950); **12**, 1–47 (1953); ces Rev. **11**, 524; **14**, 656]. D'autre part, une démonstration "a priori" assez simple de la continuité des fonctionnelles analytiques linéaires avait été donnée par Caccioppoli [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **15**, 713–714 (1932)] et modifiée plus tard par Teichmüller. L'auteur la cite en disant qu'elle a été obtenue sous des hypothèses différentes de celles de la théorie de Fantappiè. Mais la vérité c'est que la démonstration de Caccioppoli est immédiatement adaptable à la systématisation de Fantappiè. *J. Sebastião e Silva* (Lisbonne).

Korovkin, P. P. On convergence of linear positive operators in the space of continuous functions. Doklady Akad. Nauk SSSR (N.S.) **90**, 961–964 (1953). (Russian)

Let $\phi_n(x, y)$, $n=1, 2, \cdots$, be a function defined in the square R $(a \leqq x, y \leqq b)$, non-decreasing in y for every fixed x. Let

$$L_n(f) = \int_a^b f(y)d\phi_n(x, y) = f_n(x).$$

Then 1) if for $f=1, y, y^2$ the integral $L_n(f)$ tends uniformly to f on (a, b), the same holds for every continuous function f on (a, b); 2) (stated without proof) if f_0, f_1, f_2 form a Chebyshev system of functions (i.e., if $a_0f_0+a_1f_1+a_2f_2$ has no more than two zeros on (a, b), whatever the constants a_0, a_1, a_2 not simultaneously zero), and if $L_n(f_k)$ tends uniformly to f for $k=0, 1, 2$, then $L_n(f)$ tends uniformly to f for every f continuous on (a, b); 3) the conclusion of the preceding theorem fails if f_0, f_1, f_2 do not form a Chebyshev system; 4) an analogue of 2) holds for periodic functions; 5) if the numbers $\rho_k^{(n)}$ $(n=1, 2, 3, \cdots;\ k=0, 1, \cdots, n)$ satisfy the conditions

$$\rho_0^{(n)} \to 1, \quad \rho_1^{(n)} \to 1, \quad \tfrac{1}{2}\rho_0^{(n)} + \sum_{k=1}^{n} \rho_k^{(n)} \cos kx \geqq 0,$$

then for every continuous

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_1^n (a_k \cos kx + b_k \sin kx)$$

the operators

$$L_n(f) = \tfrac{1}{2}\rho_0^{(n)}a_0 + \sum_{k=1}^{n} \rho_k^{(n)}(a_k \cos kx + b_k \sin kx)$$

tend uniformly to f. *A. Zygmund.*

Kotelyanskiĭ, D. M. On an arithmetic method of orthogonalization in nonseparable Hilbert spaces. Mat. Sbornik N.S. **33(75)**, 181–192 (1953). (Russian)

If x, y are real, $x, y>0$, x/y integral, y is called a generalized divisor of x and x a generalized multiple of y, $(y \| x)$. From these the concepts of greatest common divisor $D(x, y)$ and least common multiple $M(x, y)$ are easily defined. If $\mu(n)$ is the Möbius function then (Basic Lemma) for $\delta, r, \nu>0$, $\nu \nmid r$, $\sum \mu(\nu/d)=0$, where the sum is extended over all d for which $d \| \nu$ and $D(d, r)=\delta$. This lemma is applied to the construction of orthonormal systems in a Hilbert space H for which $\dim H \geqq 2^{\aleph_0}$, i.e., H is assumed to contain an orthonormal system $\{\alpha_\nu\}$ $(0 \leqq \nu < \infty)$. A system $\{f_\nu\}$ in H fulfills the D-condition in case $(f_\mu, f_\nu)=0$ for μ/ν irrational and $(f_\mu, f_\nu)=g(D(\mu, \nu))$ for μ/ν rational, where $g(x)$ is an arbitrary function defined for $x>0$. Finally, let $w(x_1x_2)=w(x_1)w(x_2)$ if $x_1, x_2>0$ and assume

$$\sigma = \sum_1^\infty |w(k)|^2 < \infty.$$

Then the functions $\varphi_\nu = f_\nu/\sigma^{1/2}|w(\nu)|^2$ where

$$f_\nu = \sum_{k=1}^{\infty} w(\nu k)\alpha_{\nu k}$$

fulfill the D-condition (with $g(x)=|w(x)|^{-2}$). Let

$$\psi_\nu = \sum_{d \mid \nu} \mu(\nu/d)\varphi_d.$$

Then $\{\beta_\nu = \sigma^{1/2}|w(\nu)|\psi_\nu\}$ forms a new orthonormal system. The subspaces spanned by $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ are identical.

By setting $w(n)=n^s$, $\mathrm{Re}(s)>\frac{1}{2}$, $\sigma_{2s}(p/\delta)=\zeta(2s)\prod(1-\pi^{-2s})$ ($\zeta(s)$, the Riemann ζ-function; π prime, $\pi \mid p/\delta$) $=\sum k^{-2s}$ (k relatively prime to p/δ), and $\varphi_s(m)=m^s\prod_{\pi \mid m}(1-\pi^{-s})$, the identities:

$$\frac{1}{\zeta(2s)} \cdot \frac{1}{n^s m^s} \sum_{p=1}^{\infty} \frac{\varphi_{2s}(p)\mu(p/\delta)\mu(p/\delta')}{\varphi_{2s}(p/\delta)\varphi_{2s}(p/\delta')} = \delta_{mn}$$

are derived, where $\delta=D(n, p)$, $\delta'=D(m, p)$, and δ_{mn} is the Kronecker function. Another application is made to the theory of almost periodic functions. *B. R. Gelbaum.*

Kilpi, Yrjö. Über lineare normale Transformationen im Hilbertschen Raum. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **154**, 38 pp. (1953).

The author studies conditions under which a normal transformation (in the sense of Nakano) has a hypermaximal normal extension. The techniques are similar to the ones that were successfully used by von Neumann in the symmetric case. *P. R. Halmos* (Chicago, Ill.).

Rothe, E. H. Correction to the paper "Leray-Schauder index and Morse type numbers in Hilbert space". Ann. of Math. (2) **58**, 593–594 (1953).

See same Ann. (2) **55**, 433–467 (1952); these Rev. **14**, 185.

Sasaki, Siro. A proof of the spectral theorem. Kumamoto J. Sci. Ser. A. **1**, no. 2, 14–16 (1953).

This paper deals with a general form of the spectral theorem for unbounded operators. Let A be a uniformly closed commutative B^*-algebra of bounded linear operators in a separable Hilbert space $\mathfrak{H}$. Let $C(A)$ denote the set of all closed operators which commute with all those bounded operators which commute with the operators of A. Let Ω be the compact Hausdorff space of multiplicative continuous linear functionals on A (Ω is called the spectrum of A). The first main theorem is the assertion that if $U \in C(A)$ and the domain of U is everywhere dense in $\mathfrak{H}$, then there exists a measurable function $f(\chi)$ on Ω such that $(Ux, y)=\int_\Omega f(\chi)d\mu_{x,y}(\chi)$ for every $x \in \mathfrak{D}(U)$ and every $y \in \mathfrak{H}$. Here $\mu_{x,y}(\chi)$ is a measure on Ω such that, for each $T \in A$, $(Tx, y)=\int_\Omega \chi(T)d\mu_{x,y}(\chi)$. The proof hinges on finding an $x \in \mathfrak{D}(U)$ such that, for every $y \in \mathfrak{H}$, $\mu_{x,y}$ is absolutely continuous with respect to $\mu_{x,x}$. The separability of $\mathfrak{H}$ enters here, but the argument for the existence of such an x is left to the reader.

Suppose now that U is an unbounded self-adjoint operator. Let A be the maximal commutative B^*-algebra which commutes with U. It is asserted that $U \in C(A)$, so that the previous theorem is applicable to give a representation of U, and it is further indicated that $\mu_{x,y}(\chi)$ takes the form $(E(\chi)x, y)$ in this case. Not a single word of proof or explanation is given for this theorem. This is regrettable, in the opinion of the reviewer. *A. E. Taylor.*

Rosenberg, Alex. The number of irreducible representations of simple rings with no minimal ideals. Amer. J. Math. **75**, 523–530 (1953).

The following two classes of simple rings are shown to have infinitely many nonisomorphic irreducible representations. (1) If V is a vector space of Hamel dimension $\aleph \geqq \aleph_0$ over a division ring D of cardinality $\leqq \aleph$; if L is the ring of all linear transformations of V into itself and F is its maximal ideal of all linear transformations whose range has dimension $<\aleph$; then $L-F$ has exactly $2^{(2^\aleph)}$ nonisomorphic irreducible representations. A similar result is true of the ring of all bounded operators on a Hilbert space modulo its ideal of completely continuous operators. (2) If A is a

simple Zorn algebra [i.e., every nonnil right ideal contains a nonzero idempotent—see I. Kaplansky, Trans. Amer. Math. Soc. 68, 62–75 (1950); these Rev. 11, 317] of dimensional $\aleph_0$ with unit and without one-sided minimal ideals, then A has $\mathfrak{c}$ nonisomorphic irreducible representations. The proofs are based on an upper bound on the number $\mathfrak{a}$ of maximal right ideals of the ring R giving rise to isomorphic irreducible right modules (viz., $\mathfrak{a} \leqq$ cardinal number of R) and proceed by essentially showing that there is a large number of maximal right ideals. A concluding section solves a problem raised by M. A. Naĭmark [Uspehi Matem. Nauk (N.S.) 6, no. 6(46), 160–164 (1951); these Rev. 13, 755] by proving that an irreducible C^*-algebra of bounded operators on a separable Hilbert space H having a unique irreducible representation is the algebra of all completely continuous operators on H. The proof is based on a result of Naĭmark's in the paper cited above and several results of Kaplansky's [Duke Math. J. 16, 399–418 (1949); these Rev. 11, 115] and Segal's [Bull. Amer. Math. Soc. 53, 73–88 (1947); these Rev. 8, 520]. *G. K. Kalisch.*

Umegaki, Hisaharu. Operator algebra of finite class. II. Kōdai Math. Sem. Rep. **1953**, 61–63 (1953).

This extends somewhat earlier results of the author [same Rep. **1952**, 123–129; these Rev. **14**, 991] and completes their proofs. It also treats the relation between (finite numerical) traces and "semi-traces" in certain operator algebras. *I. E. Segal* (New York, N. Y.).

Turumaru, Takasi. On the direct-product of operator algebras. II. Tôhoku Math. J. (2) **5**, 1–7 (1953).

The direct product of C^*-algebras as defined by the author in part I [same J. (2) **4**, 242–251 (1952); these Rev. **14**, 991] is shown to be identical with the C^*-algebra generated by the direct products of operators in the respective algebras. The direct product of C^*-algebras with units is simple if and only if each of the factors is simple.

I. E. Segal (New York, N. Y.).

Theory of Probability

***Unkovskiĭ, V. A. Teoriya veroyatnostei.** [Theory of probability.] Voenno-Morskoe Izdat., Moscow, 1953. 320 pp. 10.90 rubles.

Mantel, Nathan. An extension of the Buffon needle problem. Ann. Math. Statistics **24**, 674–677 (1953).

Prohorov, Yu. V. Probability distributions in functional spaces. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 165–167 (1953). (Russian)

The author considers sequences of probability measures on the Banach space B of continuous functions on a compact interval. A convergent sequence is one which converges to a measure P on every set whose boundary has P measure 0. A set of measures is compact if and only if to every $\epsilon>0$ corresponds a compact B set on which all the given measures have values $\geqq 1-\epsilon$. Using this fact, the author remarks that one can obtain simple proofs of the convergence of distributions involved in a sequence of stochastic processes to the corresponding distribution of the limit process, if the finite-dimensional distributions converge. As an example he remarks that a theorem of Donsker [Mem. Amer. Math. Soc. no. 6 (1951); these Rev. **12**, 723] can be obtained in this way. *J. L. Doob* (Urbana, Ill.).

Bergström, Harald. On distribution functions with a limiting stable distribution function. Ark. Mat. **2**, 463–474 (1953).

The author applies his method [Skand. Aktuarietidskr. **34**, 1–34 (1951); these Rev. **13**, 258] to the convergence of the nth convolution of a d. f. F to a stable d. f. G_α of exponent α and obtains estimates of the remainder. The general result is rather complicated, but a particular case of interest is the following. If either (1) $0<\alpha<1$ and

$$(*) \qquad \int_{-\infty}^{\infty} |x|^{\alpha+\epsilon} d(F-G) < \infty$$

for some ϵ, $0<\epsilon<1-\alpha$; or (2) if $1 \leqq \alpha < 2$, (*) holds for some ϵ, $0<\epsilon<2-\alpha$, and $\int_{-\infty}^{\infty} x d(F-G)=0$; then

$$F^{*n}(x)=G^{*n}(x)+o(n^{-\epsilon/\alpha}).$$

The condition (*) may be weakened by replacing $|x|^{\alpha+\epsilon}$ with $|x|^{\alpha}(\log|x|)^{p}$, $p>0$; then the remainder term is correspondingly weakened to $o((\log n)^{-p})$. It should be remarked that the Fourier transform method for the normal case has failed heretofore to yield a satisfactory result in the stable case. Regrettably the style of this paper is less than lucid.

K. L. Chung (Syracuse, N. Y.).

Kendall, David G., and Rankin, R. A. On the number of points of a given lattice in a random hypersphere. Quart. J. Math., Oxford Ser. (2) **4**, 178–189 (1953).

In connection with some research on cutaneous sensations of warmth due to G. H. Wright, the authors have undertaken an investigation of the number of points of a given lattice in a random hypersphere, with particular reference to hexagonal lattices. They derive explicit formulae for the expected number of points and for the variance and compare the numerical results obtained in this way with 23,000 random throwings of circles of various sizes on to a hexagonal lattice performed by Wright. The results agree well.

R. Bellman (Santa Monica, Calif.).

Rényi, Alfréd, et Takács, Lajos. Sur les processus d'événements dérivés par un processus de Poisson et sur leurs applications techniques et physiques. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 139–146 (1953). (Hungarian. Russian and French summaries)

The first author proved earlier [Publ. Math. Debrecen **2**, 66–73 (1951); Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **1**, 202–212 (1951); these Rev. **13**, 51, 958] that the number of happenings inaugurated by the events of a Poisson process forms also a Poisson process. A new proof of this theorem is given which is based on a limit theorem for Poisson convergence of sequences of sums of random variables. *E. Lukacs* (Washington, D. C.).

Moran, P. A. P. The random division of an interval. III. J. Roy. Statist. Soc. Ser. B. **15**, 77–80 (1953).

This paper is concerned with the distribution of the variable $S=(t_1^2+\cdots+t_n^2)/(t_1+\cdots+t_n)^2$ where the t_k are mutually independent positive random variables with $\Pr\{t_k>t\}=e^{-t}$. [The problem is connected with tests for Poisson processes treated in part II, same J. **13**, 147–150 (1951); these Rev. **13**, 667. Part I appeared in Suppl. J. Roy. Statist. Soc. **9**, 92–98 (1947); these Rev. **9**, 291.] Numerical values for the 5 and 1 percent levels are given up to $n=9$ and 10, and general approximation formulae are derived useful for $n \geqq 21$. *W. Feller.*

Gnedenko, B. V. On the role of the maximal summand in the summation of independent random variables. Ukrain. Mat. Žurnal **5**, 291–298 (1953). (Russian)

Let η_n, ζ_n, s_n be respectively the maximum, maximum modulus, sum, of the first n of a sequence of mutually independent random variables with a common distribution function F. The possible limiting distributions (properly normalized) of η_n, ζ_n, s_n are known. In the present paper the author proves several theorems on the mutual relations of these limiting distributions, and their relation to F. The following results are typical. (1) If η_n, when properly normalized, has limit distribution function $\exp(-e^{-x})$, then $\int_0^\infty |x|^\delta dF(x) < \infty$ for all $\delta > 0$. (2) If ζ_n, when properly normalized, has the limit distribution in (1) or one of a second specified type, then s_n, when properly normalized, has a normal limit distribution. The possible limiting distributions of properly normalizd η_n, ζ_n, when a normalized s_n has a stable non-normal limit distribution, are found. Such results can be generalized by considering the kth largest instead of the largest member in defining η_n and ζ_n, and one such result is stated. *J. L. Doob.*

Jensen, Arne. Markoff chains as an aid in the study of Markoff processes. Skand. Aktuarietidskr. **36**, 87–91 (1953).

The Markoff process with a finite number of states defined by $P'(t) = AP(t)$, $P(0) = E$, with $P(t)$ the matrix of transition probabilities, $P'(t)$ its derivative, $A = (a_{ij})$ a matrix and E the unit matrix, is related to a Markoff chain by the representation: $P(t) = \sum B_n e^{-\alpha t}(\alpha t)^n/n!$, since $B_n = (\alpha^{-1}A + E)^n$. Here α is any number not less than the greatest of $-a_{ii}$. This is suggested for the simplification of statistical experiments common when A is not simple, as in many telephone traffic problems. *J. Riordan.*

Franckx, E. La génération d'une chaîne de Markoff. Mitt. Verein. Schweiz. Versich.-Math. **53**, 145–151 (1953).

It is noticed that the transition matrices, being powers of a finite square matrix, have certain elementary properties from the vectorial standpoint. *K. L. Chung.*

Bobrov, A. A. On determination of the order of stochastic growth of sums of random quantities. Ukrain. Mat. Žurnal **4**, 393–398 (1952). (Russian)

Let $\{X_k\}$ be a sequence of mutually independent random variables, $S_n = X_1 + \cdots + X_n$, and $F_n(x) = \Pr\{S_n \leq x\}$. Finally let M_n be the median of S_n. The condition that there exist two sequences of numbers a_k, b_k such that $F_{n_k}(a_k x + b_k) \to \Phi(x)$ where $\Phi(x)$ is a non-unitary distribution function, is shown to be equivalent to the condition that there exists a sequence B_n for which the variables $\xi_n = (S_n - M_n)/B_n$ do not converge in probability either to zero or to infinity. *W. Feller* (Princeton, N. J.).

Pugačev, V. S. The general theory of correlation of random functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 401–420 (1953). (Russian)

The author discusses (purely formally) stochastic processes and their transformations under linear and non-linear operations, from the point of view of representations of these processes by means of series of functions with orthogonal random coefficients. [For these representations see Karhunen, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 37 (1947); these Rev. **9**, 292.] *J. L. Doob.*

Yaglom, A. M., and Pinsker, M. S. Random processes with stationary increments of the nth order. Doklady Akad. Nauk SSSR (N.S.) **90**, 731–734 (1953). (Russian)

The authors study continuous-parameter stochastic processes whose nth order differences are stationary in the wide sense, finding the spectral representation of such a process and the corresponding form of the covariance function. The case $n = 1$ was treated by Kolmogorov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 6–9, 115–118 (1940); these Rev. **2**, 220]. *J. L. Doob* (Urbana, Ill.).

Łopuszański, Jan. Relativisierung der Theorie der stochastischen Prozesse. Acta Phys. Polonica **12**, 87–99 (1953). (Russian summary)

It has often been remarked that the assumptions underlying the diffusion equation require infinite velocities. [This, of course, is not true of the refined Uhlenbeck-Ornstein model which considers the process in the phase space.] In relativity one should require that a transition from x to y is possible only if $|x - y| < ct$. The author discusses formally how the Chapman-Kolmogorov identity can be written invariantly. However, when it comes to the diffusion equation his simple conclusion seems to be that the covariance tensor must be zero, that is, that the diffusion equation must reduce to the deterministic limiting form (in the case of one dimension to $u_t = au_x$). This, of course, does not contradict relativity but amounts to an abolition rather than a relativization of diffusion processes. *W. Feller.*

Hinčin, A. Ya.* The concept of entropy in the theory of probability. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 3–20 (1953). (Russian)

Detailed treatment of the elementary definitions and inequalities of information theory. A fuller use of the law of large numbers, as applied to stationary Markov chains, would have eliminated some of the calculations.

J. L. Doob (Urbana, Ill.).

Mandelbrot, Benoit. Contribution à la théorie mathématique des jeux de communication. Publ. Inst. Statist. Univ. Paris **2**, nos. 1–2, 124 pp. (1953).

The author undertakes to study problems of communication from the point of view of the mathematical theory of games of strategy involving three players: the sender, Nature, and the receiver, with various conditions among the players. He points out that the strategic framework can be used directly as in the work of Shannon [Bell System Tech. J. **27**, 379–423, 623–656 (1948); these Rev. **10**, 133] on the technical problems of transmission given the statistical properties of messages and the physical properties of Nature. He asserts that, equally, the inverse approach can be used: one can construct perfect plays and show that the results are verified by physical phenomena. Two examples are considered in detail; thermodynamics as an optimal minimax game between sender and Nature and the statistical structure of language in relation to a minimum coalition of sender and receiver. *S. Sherman.*

Arfwedson, G. Research in collective risk theory. The case of equal risk sums. Skand. Aktuarietidskr. **36**, 1–15 (1953).

Let x be the net risk premium and λ the security loading so that the total premium received by the risk fund is λx. Denote by u the initial value of the risk fund and by $G(x, u)$ the probability of ruin at or before a time when a net premium x has flowed into the risk fund. The author assumes

that the sums at risk are equal and derives formulae for the probability $F(x, u)=1-G(x, u)$ for values of x and u for which λx and u are integers. *E. Lukacs.*

Tedeschi, B. Alcune considerazioni sulla teoria classica e sulla teoria collettiva del rischio. Giorn. Ist. Ital. Attuari **14** (1951), 16–35 (1952).

The paper is a contribution to the polemic discussion on risk theory with special reference to the papers by Ottaviani [same Giorn. **11**, 163–189 (1940); these Rev. **8**, 390] and Segerdahl [Skand. Aktuarietidskr. **25**, 43–83 (1942); these Rev. **8**, 215]. The author prefers the classical theory of risk to the collective theory of risk since in his opinion the collective theory of risk is based on unrealistic assumptions. *E. Lukacs* (Washington, D. C.).

Mathematical Statistics

***Zia-ud-Din, M., and Moin-ud-Din Siddiqi, M. Random sampling numbers.** Proceedings of The Second Pakistan Statistical Conference, University of Dacca, January, 1952, pp. 91–98. Lahore, 1953.

A table of 8000 random sampling numbers was constructed using two decks of playing cards with kings, queens, jacks removed. After shuffling, one card was drawn and its number noted, 10 serving as 0. *S. W. Nash.*

Barberi, Benedetto. Statistica e calcolo delle probabilità. Statistica, Bologna **13**, 139–162 (1953).

Sitgreaves, Rosedith. On the distribution of two random matrices used in classification procedures. Ann. Math. Statistics **23**, 263–270 (1952).

Certain statistics, U and W, have been proposed by A. Wald and T. W. Anderson respectively, as solutions for the problem of classifying a multivariate normal population π as being either π_1 or π_2. The populations π_i have unknown means $\mu^{(i)}$ $(i=1, 2)$ and a common unknown covariance matrix Σ. The decision is based on N_1+N_2+1 independent p-dimensional observations, N_i from π_i and 1 from π. The statistics U and W are special cases of certain related 2-by-2 matrices, M and M^*. In this paper an analytic derivation of the distribution of M and M^*, including the constant of the distribution, is given under a certain restriction. The structure of U is $Y_1'A^{-1}Y_2$ where Y_i are independent multivariate normal variables with means $\nu^{(i)}$ $(i=1, 2)$ and common covariance matrix Σ and A^{-1} is a symmetric matrix with a Wishart distribution obtained from a parent population which is independent of Y_i $(i=1, 2)$ and has covariance matrix Σ. The restriction mentioned above is that $\nu^{(1)}$ and $\nu^{(2)}$ are proportional. *M. Sobel* (Ithaca, N. Y.).

Beall, Geoffrey, and Rescia, Richard R. A generalization of Neyman's contagious distributions. Biometrics **9**, 354–386 (1953).

The distributions having characteristic functions

$$\varphi(u)=\exp\left\{-m_1+m_1 n\int_0^1 e^{ux}(1-x)^{n-1}dx\right\}$$

with $u=m_2[e^{it}-1]$ form a three-parameter family $\mathfrak{F}$ containing Neyman's contagious distributions of type A, B, C [Neyman, Ann. Math. Statistics **10**, 35–57 (1939)]. The authors give recurrence formulae for the computation of integrals of the form $\int_0^1 e^{-mx}x^k(1-x)^{n-1}dx$ from which the frequency functions of the class $\mathfrak{F}$ can be obtained. These distributions are then fitted by the method of moments to numerous sets of experimental data. The goodness of fit is measured by the usual χ^2. It seems to the reviewer that the probabilities associated to these χ^2 should be interpreted with caution since there is no assurance of independence of the observations. *L. LeCam.*

Silvey, Samuel D. The equivalence of asymptotic distributions under randomisation and normal theories. Proc. Glasgow Math. Assoc. **1**, 139–147 (1953).

Let $t_n=t_n(x_1, x_2, \cdots, x_n)$ be independent of changes in location and scale. The author is interested in finding conditions under which the distribution of t_n under randomization over an arbitrary set of numbers $a_1, a_2, \cdots, a_n$ with $a_i \neq a_j$ for some i, j is asymptotically equivalent to the distribution of t_n under random sampling from some normal distribution. If t_n is linear in the (standardized) x_i, a necessary and sufficient condition for asymptotic equivalence is that the distribution of the set of a_i tend to the normal distribution. The author generalizes a theorem of the reviewer [Ann. Math. Statistics **20**, 455–458 (1949); these Rev. **11**, 188] on the limiting distribution of a linear form under randomization. He seems to be unaware that his formulation can be obtained from a theorem of Hoeffding [ibid. **22**, 558–566 (1951); these Rev. **13**, 363]. *G. E. Noether.*

Romanovskiĭ, V. I. On an implicit lemma and lemmas similar to it. Doklady Akad. Nauk UzSSR **1949**, no. 4, 3–5 (1949). (Russian)

The author shows how the large-sample limiting distributions of functions of sample means can be obtained by the use of the central limit theorem and appropriate transformation theorems. P. L. Hsu [Proc. Berkeley Symposium on Math. Statistics and Probability, 1945, 1946, Univ. of California Press, 1949, pp. 359–402; these Rev. **10**, 387] has given a detailed treatment of this method, generalizing results of the reviewer [Ann. Math. Statistics **6**, 160–169 (1935)]. The author also generalizes the latter results, and is evidently not cognizant of the former ones. *J. L. Doob* (Urbana, Ill.).

Scheffé, Henry. A method for judging all contrasts in the analysis of variance. Biometrika **40**, 87–104 (1953).

A simple answer is found for the following question which has plagued the practice of the analysis of variance: Under the usual assumptions, if the conventional F-test of the hypothesis H: $\mu_1=\mu_2=\cdots=\mu_k$ at the α level of significance rejects H, what further inferences are valid about the contrasts among the μ_i (beyond the inference that the values of the contrasts are not all zero)? Suppose the F-test has $k-1$ and ν degrees of freedom. For any $c_1, \cdots, c_k$ with $\sum_1^k c_i=0$ write θ for the contrast $\sum_1^k c_i\mu_i$, and write $\hat{\theta}$ and $\hat{\sigma}^2_{\hat{\theta}}$ for the usual estimates of θ and the variance of $\hat{\theta}$. Then for the totality of contrasts, no matter what the true values of the θ's, the probability is $1-\alpha$ that they all satisfy $\hat{\theta}-S\hat{\sigma}_{\hat{\theta}} \leq \theta \leq \hat{\theta}+S\hat{\sigma}_{\hat{\theta}}$, where S^2 is $(k-1)$ times the upper α point of the F-distribution with $k-1$ and ν degrees of freedom. Suppose we say that the estimated contrast $\hat{\theta}$ is "significantly different from zero" if $|\hat{\theta}|>S\hat{\sigma}_{\hat{\theta}}$. Then the F-test rejects H if and only if some $\hat{\theta}$ are significantly different from zero, and if it does, we can say just which $\hat{\theta}$. More generally, the above inequality can be employed for all the contrasts with the obvious frequency interpretation about

the proportion of experiments in which all statements are correct. Relations are considered to an earlier method of Tukey using the Studentized range tables and valid in the special case where the $\hat{\mu}_i$ all have the same variance and all pairs $\hat{\mu}_i$, $\hat{\mu}_j$ $(i \neq j)$ have the same covariance. Some results are obtained for the operating characteristic of the new method. (From the author's summary.)

Z. W. Birnbaum (Seattle, Wash.).

Cansado, Enrique. Expectations and variances in multi-stage sampling. Trabajos Estadistica **3**, 27–41 (1952). (Spanish summary)

This is an expository article aimed at amplifying the proofs in Deming, "Some theory of sampling" [Wiley, New York, 1950, chap. 5]. *B. Epstein* (Detroit, Mich.).

Proschan, Frank. Confidence and tolerance intervals for the normal distribution. J. Amer. Statist. Assoc. **48**, 550–564 (1953).

Confidence and tolerance intervals for the normal distribution are presented for the various cases of known and unknown mean and standard deviation. Tables permit comparison among the intervals. One table gives the factor $k_{\alpha,\alpha}$ for tolerance intervals such that $\bar{x} \pm k_{\alpha,\alpha}$ will include a proportion α of the population on the average, for $\alpha = .50, .75, .90, .95, .98, .99, .999$ and sample size $n = 2(1)30, 40, 60, 120, \infty$. An appendix gives a proof of Wilks' formula [Ann. Math. Statistics **12**, 91–96 (1941); these Rev. **3**, 9] for tolerance limits including a proportion α of the population on the average. *S. W. Nash* (Vancouver, B. C.).

Walsh, John E. Large sample confidence intervals for density function values at percentage points. Sankhyā **12**, 265–276 (1953).

Let $x(1), \cdots, x(n)$ denote sample values from a population with density $f(x)$, arranged in order of increasing magnitude. Let θ_p be determined by $\int_{-\infty}^{\theta_p} f(x)dx = p$. The author obtains approximate confidence intervals for $f(\theta_p)$ by what is essentially an application of the mean-value theorem, with the empirical cumulative distribution and order statistics used in place of their theoretical counterparts. Specifically, the statistic used is

$$2\epsilon/\{x[(p+\epsilon)n + c\sqrt{n}] - x[(p-\epsilon)n - c\sqrt{n}]\},$$

where c_α is a function of the standard normal deviate exceeded with probability α. Various restrictions are made to improve the accuracy of the approximate result; e.g., $0.05 \leq p \leq 0.95$, $f(x)$ must satisfy some mild restrictions. Lower bounds are given for n, and upper and lower bounds for the confidence coefficient. *S. W. Nash.*

Sengupta, J. M. Significance level of $\sum x^2/(\sum x)^2$ based on Student's distribution. Sankhyā **12**, 363 (1953).

Aoyama, Hirojiro. On the chi-square test for weighted samples. Ann. Inst. Statist. Math., Tokyo **5**, 25–28 (1953).

Meier, Paul. Variance of a weighted mean. Biometrics **9**, 59–73 (1953).

Let $\bar{\mu}$ be a weighted mean of k independent normal unbiased estimates of a parameter μ having variances $\sigma_1^2, \cdots, \sigma_k^2$, where the weights are proportional to the reciprocals of independent unbiased estimates of $\sigma_1^2, \cdots, \sigma_k^2$, respectively, having mean square distributions with $n_1, \cdots, n_k$ degrees of freedom. The author gives an asymptotic expression for $V(\bar{\mu})$ (the variance of $\bar{\mu}$), an approximately unbiased estimate of $V(\bar{\mu})$, and an approximation to the distribution of this estimate. For $k = 2$ the author gives bounds on $V(\bar{\mu})$ and, in special cases, exact formulas for $V(\bar{\mu})$.

D. F. Votaw, Jr. (New Haven, Conn.).

Jones, Howard L. Approximating the mode from weighted sample values. J. Amer. Statist. Assoc. **48**, 113–127 (1953).

The author derives weights for estimation of the mode of a distribution by means of a linear systematic statistic and gives examples of computation of the weights. [Systematic statistics are discussed in Mosteller, Ann. Math. Statistics **17**, 377–408 (1946); these Rev. **8**, 477.] A table of weights is given for samples of size 3(1)10 from a t-distribution for the following values of the kurtosis: 3(.5)5, 6, 9, ∞.

D. F. Votaw, Jr. (New Haven, Conn.).

Yoneda, Keizo. On the use of the Neyman's allotation. Yokohama Math. J. **1**, 117–123 (1953).

For choosing, in stratified sampling, between (i) sample sizes proportionate to relative stratum sizes P_i, assumed known, and (ii) sample sizes proportionate to products $P_i s_i$, the s_i^2 being pilot sample estimates of the unknown stratum variances σ_i^2, a rule is given, which assumes that "upper and lower limits" of $\sum P_i \sigma_i / (\sum P_i \sigma_i^2)^{\frac{1}{2}}$ are available.

D. M. Sandelius (Uppsala).

Lieberman, Gerald J. A note on Dodge's continuous inspection plan. Ann. Math. Statistics **24**, 480–484 (1953).

In H. F. Dodge's procedure for continuous sampling inspection [same Ann. **14**, 264–279 (1943); these Rev. **5**, 130] all items are inspected till i in succession are found free of defects, then one item out of each k items is inspected until a defective item is found, when the whole procedure begins again with inspection of all items. All defective items found are replaced by good items. The present author proves that, with probability one, the limit superior of the proportion of defective items in the outgoing product $\leq (k-1)/(k+i)$ as the process continues indefinitely, even when the production process is not in a state of statistical control. The upper bound $(k-1)/(k+i)$ is attained if the process gives only non-defective items during total inspection, only defective items during partial inspection. *S. W. Nash.*

Giesekus, Hanswalter. Die Anwendung der statistischen Prüfverfahren auf Reihen mit Erhaltungsneigung und kontinuierliche Gesamtheiten. Mitteilungsblatt Math. Statist. **5**, 103–124 (1 plate) (1953).

The paper deals with the estimation of the mean, the variance and the correlation coefficient in a stationary sequence of dependent random variables (or random vectors) by means of the corresponding sample moments.

W. Hoeffding (Chapel Hill, N. C.).

Binet, F. E. The fitting of the positive binomial distribution when both parameters are estimated from the sample. Ann. Eugenics **18**, 117–119 (1953).

The problem indicated, which has been considered by several authors, is reviewed in the light of the recent work of Hammersley on estimating restricted parameters [J. Roy. Statist. Soc. Ser. B. **12**, 192–240 (1950); these Rev. **12**, 725]. *D. G. Chapman* (Seattle, Wash.).

Des Raj. On moment estimation of the parameters of a normal population from singly and doubly truncated samples. Ganita **4**, 79–84 (1953).

Let X have normal distribution $N(\mu, \sigma^2)$ with unknown μ, σ^2, and let $X_0' < X_0''$ be two known numbers. The following situations are considered: (a) X is truncated to (X_0', X_0''), i.e., to arrive at a sample of size n, observations of X are obtained until n of them are in the interval (X_0', X_0''), while observations outside of (X_0', X_0'') are disregarded; (b) X is censored in each tail separately, i.e., for a sample of total size n, the values falling in (X_0', X_0'') and their frequency n_0 are recorded, while values below X_0' and above X_0'' are only counted and their frequencies n_1 and n_2 are recorded, so that $n_1+n_0+n_2=n$; c) X is censored in both tails jointly, i.e., for a sample of size n, the values falling in (X_0', X_0'') and their frequency n_0 are recorded, while values outside (X_0', X_0'') are only counted and their total frequency $n-n_0$ is recorded. For each of these cases the author shows that estimates of μ and σ^2 obtained by the method of moments (subject to a plausible modification in cases (b) and (c)) are the same as the maximum likelihood estimates obtained by Cohen [Ann. Math. Statistics **21**, 557–569 (1950); these Rev. **12**, 346] and Hald [Skand. Aktuarietidskr. **32**, 119–134 (1949); these Rev. **12**, 193]. This is a generalization of a result obtained by R. A. Fisher for the case of one-sided truncation [Circular and hyperbolic functions . . . , British Association for the Advancement of Science, Mathematical Tables, vol. 1, Cambridge, 1931, pp. xxvi–xxxv]. *Z. W. Birnbaum.*

Des Raj. On estimating the parameters of bivariate normal populations from doubly and singly linearly truncated samples. Sankhyā **12**, 277–290 (1953).

Let $f(X, Y; m_x, m_y, \sigma_x^2, \sigma_y^2, \rho)$ be the probability density of a bivariate normal population π. For given $k_1 < k_2$, the author defines the population π^* with the probability density $f^* = f \cdot (\int_{-\infty}^{+\infty}\int_{k_1}^{k_2} f\,dY\,dX)^{-1}$ for $k_1 < Y < k_2$, and $f^* = 0$ elsewhere, as the population π doubly truncated in Y to the set $k_1 < Y < k_2$. Extending his results for univariate populations [see the paper reviewed above], the author considers the cases of strictly doubly truncated populations, populations censored for $Y \leqq k_1$ and for $Y \geqq k_2$ separately, and populations censored for $Y \leqq k_1$ and $Y \geqq k_2$ jointly. For each case he derives systems of equations which determine estimates of the five parameters of π, both by the method of moments and the method of maximum likelihood, and shows that both methods yield the same results.

Z. W. Birnbaum (Seattle, Wash.).

Chernoff, Herman. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. Ann. Math. Statistics **23**, 493–507 (1952).

As opposed to the classical Neyman-Pearson approach to the problem of testing one simple hypothesis H_0 against another H_1, the author considers minimizing $\beta+\lambda\alpha$ where β, α are the traditional Type II and Type I errors of a test whose critical region is of the form $S_n = \sum_{i=1}^{n} X_i > k$ (the X_i's being independent, identically distributed random variables) and λ is a given positive number. The minimum value of $\beta+\lambda\alpha$ is roughly ρ^n, the "index" ρ being independent of λ. This suggests utilizing $\log \rho_1 / \log \rho_2$ as a measure of the relative efficiency of two tests having indices ρ_1 and ρ_2. If H_0 and H_1 are "close" to one another, ρ may be approximated by assuming X_i normally distributed.

The results depend upon bounds for the "tail" probabilities of S_n in terms of the moment-generating and related functions, these relating to work of H. Cramér [Colloque consacré à la théorie des probabilités, 3ième partie, Actualités Sci. Ind., no. 736, Hermann, Paris, 1938, pp. 5–23]. In line (5: 11) the subscripts 0 should be replaced by i.

H. Teicher (Lafayette, Ind.).

Roy, S. N. On a heuristic method of test construction and its use in multivariate analysis. Ann. Math. Statistics **24**, 220–238 (1953).

If $\omega(H_0, H_i, \beta)$ is the most powerful (most powerful bisimilar) critical region of a test of the simple (composite) hypothesis H_0 against the simple (composite) hypothesis H_i, then a Type I test of H_0 against the class H_i is one whose critical region is given by $\bigcup_i \omega(H_0, H_i, \beta)$. A Type II test may be formulated similarly but is identified with the likelihood ratio test. Many classical tests are Type I as well as Type II. Each of several hypotheses concerning multivariate normal populations is transformed to a set of hypotheses about univariate (or bivariate) normal populations, the intersection of the elements of the set yielding the original (multivariate) hypothesis. This permits the calculation of a lower bound for the rather complicated power of the multivariate test in terms of the powers of the (Type I) univariate or bivariate tests. In line 2 of page 225, x should be deleted and line 20 page 221 should read $\tilde{\omega}(H_0, H_i, \beta_i)$.

H. Teicher (Lafayette, Ind.).

Kiefer, J. On minimum variance estimators. Ann. Math. Statistics **23**, 627–629 (1952).

Let x be a random variable with density $f(x, \theta)$ with respect to some fixed σ-finite measure μ ($\theta \in \Omega$, $x \in \mathfrak{X}$). For each θ let λ_1, λ_2 be any two distinct probability measures on $\Omega_\theta = \{h \mid (\theta+h) \in \Omega\}$, with finite first moment ($E_i h$, $i=1, 2$). Then for any unbiased estimate $t(x)$ of θ

$$E_\theta(t-\theta)^2 \geqq \sup \left\{ \frac{(E_1 h - E_2 h)^2}{\int_{\mathfrak{X}} \left[\left\{ \int_{\Omega_\theta} f(x; \theta+h)\, d[\lambda_1(h) - \lambda_2(h)] \right\} \Big/ f(x; \theta) \right] d\mu} \right\};$$

for each θ the supremum is taken over all λ_1, λ_2 such that the integrand of the integral over $\mathfrak{X}$ is defined a.e. (μ).

This lower bound is an improvement of one due to Chapman and Robbins [same Ann. **22**, 581–586 (1951); these Rev. **13**, 367]. A similar method of proof is used. This stronger inequality is that which (under additional restrictions) was shown by Barankin [ibid. **20**, 477–501 (1949); these Rev. **11**, 529] to be the best possible. The form given here may be more useful for applications. Some examples are given. *D. G. Chapman* (Seattle, Wash.).

Hoeffding, Wassily. The large-sample power of tests based on permutations of observations. Ann. Math. Statistics **23**, 169–192 (1952).

Tests based on permutations of the observations differ from conventional tests mainly in the fact that the critical value of the statistic to be used depends on the given observations. If the critical value converges in probability to a constant, it is possible to evaluate the large-sample power of the test in terms of the distribution of the test statistic. The main object of the paper is to indicate several methods for ascertaining that the critical value converges in probability to a constant. The results are applied to tests for the median of symmetrical distributions, analysis of variance tests, two-sample tests, and tests of randomness.

Conditions are given under which these tests based on permutations of the observations are asymptotically as powerful as the corresponding parametric tests. The following theorem generalizing the second limit theorem of probability is of interest per se: Let $F(y)$ be a distribution function on the real line determined uniquely by its moments $\mu_k=\int_{-\infty}^{\infty}y^k dF(y)$, $k=1, 2, \cdots$. Let $\{F_n(y)\}$, $n=1, 2, \cdots$, be a sequence of random distribution functions with moments μ_{nk}, and suppose that $\mu_{nk}\to\mu_k$ in probability as $n\to\infty$, $k=1, 2, \cdots$. Then $F_n(y)\to F(y)$ in probability at every point of continuity of $F(y)$. *G. E. Noether.*

Lindley, D. V. Statistical inference. J. Roy. Statist. Soc. Ser. B. **15**, 30–65; discussion, 65–76 (1953).

The author considers, in the formulation of Wald, the problem of making one of l decisions $\{d_1, \cdots, d_l\}$, when the unknown distribution is one of the set $\{f_1, \cdots, f_m\}$, and the sample size is fixed. Sections 3 and 4 of his paper are devoted to the advocacy of a method which is indistinguishable from a Bayes solution with weight functions independent of the observations.

In Section 1 the author lets $m=l$, and d_j be the decision that f_j is the actual distribution. For any decision function y let $T(y)=\{T_{ij}(y)\}$, $i, j=1, \cdots, m$, where $T_{ij}(y)$ is the probability of choosing the distribution f_j when f_i is actually the distribution and the decision function y is employed. [This vector is employed in [A] Dvoretzky, Wald, and Wolfowitz, Ann. Math. Statistics **22**, 1–21 (1951); these Rev. **12**, 515.] The author proposes to compare two decision functions y and y' on the basis of $T(y)$ and $T(y')$. In the Wald theory y and y' are compared on the basis of $r(y)$ and $r(y')$, where $r(y)=\{r_i(y)\}$, $i=1, \cdots, m$, $r_i(y)=\sum_{j=1}^{l}\int y_j(x)W_{ij}(x)df_i(x)$, $y_j(x)$ is the probability according to y of making decision d_j when x is the observed point, and $W_{ij}(x)$ is the corresponding loss function. The author proves complete class theorems and claims that his results generalize those of Wald for the present problem because, when $W_{ii}\equiv 0$ and $W_{ij}(i\neq j)$ is a positive constant, the vector $r(y)$ can be calculated from the vector $T(y)$, and a decision function admissible according to the classification by T is admissible according to the classification by r. This claim to greater generality is incorrect and actually the author's results are rather special cases of Wald's results. The following elementary device (employed in Section 3 of [A]) shows that $T(y)$ is simply $r(y)$ for a special weight function: Suppose there are m^2 possible distributions f_{ik}, $i, k=1, \cdots, m$, where the index k is fictitious and $f_{ik}\equiv f_i$, $k=1, \cdots, m$. Let W_{ij} be the (constant) loss when f_{ik} is the actual distribution and the decision d_j $(j=1, \cdots, m)$ is made. Define W_{ij} as follows: $W_{ii}=0$; $W_{ij}=1$, $j\neq i$; $W_{ik}=1$, $i\neq k$; $W_{ij}=0$, $i\neq k$, $j\neq k$. For this weight function the vector $r(y)=\{r_{ik}(y)\}$, $i, k=1, \cdots, m$, is then as follows: $r_{ik}=T_{ik}$, $i\neq k$, $r_{ii}=\sum_{k\neq i}T_{ik}$. Thus all results to be obtained by comparing T's are also to be obtained by comparing r's; the converse is not true because, e.g., in [B] Wald and Wolfowitz, Proc. 2nd Berkeley Symposium on Math. Statistics and Probability, 1950 [Univ. of California Press, 1951, pp. 149–158; these Rev. **13**, 667] where the same problem is treated, the function W is allowed to be a function of x. The author's principal complete class theorem (Theorem 2.2) is a rather special case of those of Wald [Statistical decision functions, Wiley, New York, 1950, Chapter 3; these Rev. **12**, 193]. The author seems to be unaware that he is repeating the statement and proof of Theorem 1 of [B] for the special case where W_{ij} does not depend on x. In his Theorem 2.3 he gives a sufficient condition for admissibility; this is a very special case of Theorem 2 or of Theorem 3 of [B].

The author also considers the problem of elimination of randomization from the decision function. He does not seem to be aware of the fact that his result on the elimination of randomization is simply Theorem 3.2 of [A]. By use of the simple device employed in [A] and described above, his results also follow from those of Blackwell [Ann. Math. Statistics **22**, 112–114 (1951); these Rev. **12**, 486] and Dvoretzky, Wald, and Wolfowitz [Pacific J. Math. **1**, 59–74 (1951); these Rev. **13**, 331]. *J. Wolfowitz.*

Dvoretzky, A., Kiefer, J., and Wolfowitz, J. Sequential decision problems for processes with continuous time parameter. Problems of estimation. Ann. Math. Statistics **24**, 403–415 (1953).

The authors continue the study of decision theory for stochastic processes with a continuous time which they began in an earlier paper [same Ann. **24**, 254–264 (1953); these Rev. **14**, 997]. They consider the problem of parameter estimation for the Poisson, Gamma, Normal, and Negative Binomial processes. With a proper weight function the minimax rule reduces to a fixed time rule. For example, suppose that the process being observed is a Poisson process with parameter $\lambda>0$, i.e., a process with independent stationary increments which satisfies $P\{X(t)=x\}=(\lambda t/x)e^{-\lambda t}$, $x=0, 1, 2, \cdots$, for all $t\geq 0$. A minimax estimate is desired for λ assuming a weight function $W(\lambda, \delta)=(1/\lambda)(\delta-\lambda)^2$ and a cost function $c(t)$ which is nonnegative, lower semicontinuous, and which tends to infinity as $t\to\infty$. If $x(t)$ is the observed sample curve then such an estimate is given by $x(t_0)/t_0$ where t_0 is given by

$$c(t_0)+(1/t_0)=\min_{t>0}[c(t)+(1/t)].$$

J. L. Snell (Princeton, N. J.).

Mann, Henry B. On the estimation of parameters determining the mean value function of a stochastic process. Sankhyā **12**, 117–120 (1952).

x_t is a second-order process with mean zero and $y_t=x_t+f(t)$ $(0\leq t\leq T)$, where $f(t)$ is assumed to be of the form $K_1\phi_1(t)+\cdots+K_n\phi_n(t)$, $\phi_j(t)$ $(j=1, \cdots, n)$ being a set of differentiable and independent functions. By this the author means that for any sequence of real numbers $K_1, \cdots, K_n$ not all zero, $K_1\phi_1'(t)+\cdots+K_n\phi_n'(t)\neq 0$ for a set of positive measure in $[0, T]$. The problem is to estimate the K's, given one observation of y_t over $[0, T]$. Routine calculations yield an unbiased estimate $\hat{K}_j$ of K_j, and in the case when x_t is a process of uncorrelated increments with $\sigma^2_{x_{t+\tau}-x_t}=c\tau$, the covariance matrix of $\hat{K}_1, \cdots, \hat{K}_n$. Examples are discussed.

G. Kallianpur (Calcutta).

Mathematical Biology

Reid, A. T. On stochastic processes in biology. Biometrics **9**, 275–289 (1953).

"The purposes of this article are: (i) to discuss the role of the theory of stochastic processes in the methodology of mathematical biology, (ii) to present a review of some work dealing with the application of stochastic processes in biology, and (iii) to encourage, perhaps, other workers to utilise the theory of stochastic processes in formulating

mathematical models of various biological phenomena". The author achieves his object by giving a very brief account of Markov chains and branching processes, and an interesting account of recent applications to the construction of mathematical models for (1) neural nets, (2) the depolymerisation of macromolecules by radiation damage, (3) dominance relations and "peck-right" structures in animal societies, (4) the spread of epidemics and rumours and (5) communication nets. The survey though incomplete will be very useful and it is supported by a bibliography of forty items. [The indices i and j have been interchanged at the foot of p. 277 and at the head of p. 278.]

D. G. Kendall (Oxford).

Kendall, David G. Les processus stochastiques de croissance en biologie. Ann. Inst. H. Poincaré **13**, 43–108 (1952).

In addition to summarizing most of the known results, due to the author and others, on the probabilistic theory of bacterial growth and related topics, this paper contains some new work. The point of view is mainly mathematical but a discussion is given of the biological significance of some of the usual mathematical models. The main new material is in the treatment of bacterial mutations. Both the normal and the mutant populations are allowed to follow general stochastic growth laws (previous treatments have supposed either a deterministic growth law for the normal population or a negative-exponential life-length distribution for its members). In this setting the author treats the phenomenon of a phenotypic delay in the appearance of a characteristic resistance property of the mutants. An explicit solution for the generating function of the number of resistant individuals is given in case the normal bacteria have a negative-exponential life-length distribution. (The first stochastic treatment of phenotypic delay was given by Armitage [J. Roy. Statist. Soc. Ser. B. **14**, 1–40 (1952); these Rev. **14**, 393].) Other topics treated: birth-and-death processes, immigration, interaction of species, distribution of ages. *T. E. Harris* (Santa Monica, Calif.).

Juvancz, Iréneusz, et Liptàk, Tamás. Sur quelques problèmes de l'application médico-biologique de la statistique mathématique. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 175–205 (1953). (Hungarian. Russian and French summaries)

Consider a random variable with continuous distribution function and take $2n$ independent observations forming two samples of size n. Denote by $S_n(x)$, respectively $T_n(x)$, the empirical distributions determined by the first and second sample. Let

$$D_n^+ = \sup_x [S_n(x) - T_n(x)], \quad D_n^- = \sup_x [T_n(x) - S_n(x)]$$

and

$$D_n = \sup_x |S_n(x) - T_n(x)| = \max (D_n^+, D_n^-).$$

Gnedenko and Korolyuk [Doklady Akad. Nauk SSSR (N.S.) **80**, 525–528 (1951); these Rev. **13**, 570] gave the exact distribution of the statistic D_n and Gnedenko and Rvačeva [ibid. **82**, 513–516 (1952); these Rev. **13**, 760] determined the exact joint distribution of D_n^- and D_n^+.

The authors discuss the importance of these results for statistical problems occurring in medical research and construct two tables: (a) one for the probability $P(D_n \geqq c/n)$ for $c=1(1)18$ and $n=5(1)30$; (b) a table for the probability $P(D_n^- \leqq a/n,\ D_n^+ \geqq b/n)$ for $n=4(1)30$, $b=1(1)n$ and a sequence of a values of varying length starting with $a=0$. A table equivalent to (a), namely a table for $P(D_n \leqq c/n)$, was given by F. J. Massey [Ann. Math. Statistics **22**, 125–128 (1951); these Rev. **12**, 621] for $n=1(1)40$, $c=1(1)12$. The authors also discuss in general terms statistical problems occuring in medical research and emphasize the importance of distribution-free tests of significance for small samples. In addition to the two tests for which they computed tables, they recommend a test designed by Gnedenko and Mihalevič [Doklady Akad. Nauk SSSR (N.S.) **82**, 841–843 (1952); these Rev. **13**, 760]. They also give a survey of some of the more important results in the theory of order statistics. *E. Lukacs* (Washington, D. C.).

Reid, A. T. An age-dependent stochastic model of population growth. Bull. Math. Biophys. **15**, 361–365 (1953).

Landahl, H. D. On the spread of information with time and distance. Bull. Math. Biophys. **15**, 367–381 (1953)

Reenpää, Yrjö. Über die Struktur der Sinnesmannigfaltigkeit und der Reizbegriffe. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1953**, 3–28 (1953).

The author attempts to develop a Riemannian geometry of the sensory manifold, with intensity, quality, localization, and time represented by orthogonal axes, and the discriminal thresholds providing the units of measure. The significance, either philosophic or scientific, is not clear to this reviewer. The interesting scientific questions relate to the determination of independent dimensions in quality, not attempted in this paper. *A. S. Householder.*

Karreman, George. Contributions to the mathematical biophysics of the cardiovascular system. Bull. Math. Biophys. **15**, 185–195 (1953).

The author continues his study of small pressure waves in an elastic tube filled with incompressible fluid, and of the reflection of such waves at a discontinuity of the tube diameter [same Bull. **14**, 327–340 (1952); **15**, 109 (1953); these Rev. **14**, 781]. Expressions are derived in terms of constants of the system for the ratio of the amplitudes and the phase difference of the pressure waves in front of, and behind, a constriction in the tube. Using these, a method is indicated to locate the constriction by taking the appropriate measurements. *G. W. Morgan.*

Rashevsky, N. Mathematical biology of the central nervous system with special reference to the problems of Gestalt and perception of relations. Thalès **7** (1951) 51–68 (1953).

TOPOLOGY

Calabi, L. Notions fondamentales de topologie générale. Rev. Questions Sci. (5) **14**, 393–411, 542–558 (1953).

This study is intended to facilitate access to [such] treatises of general topology where the axiomatic character and technique sometimes hides the fundamental ideas, simple and natural, on which it rests. It will expound the inductive reasonings which have led from sets of numbers to sets of elements of arbitrary nature treated in general topology. To those who will not engage at once in a deeper study, it will at least permit an understanding of the guiding ideas of a discipline basic to modern mathematics. (Author's introduction, roughly translated.) *R. F. Arens.*

Northam, E. S. The interval topology of a lattice. Proc. Amer. Math. Soc. **4**, 824–827 (1953).

The interval topology of a lattice results on taking the closed intervals as a sub-basis for the closed sets. The author shows that a necessary condition that a lattice be a Hausdorff space in its interval topology is that each closed interval have a finite separating set; this is a finite set F of elements lying between the ends of the interval, such that each element of the interval is comparable with at least one element of F.

It follows that a Boolean algebra is a Hausdorff space in its interval topology if and only if each element is over an atom. This result, which was obtained independently by Katětov [Colloquium Math. **2**, 229–235 (1952); these Rev. **14**, 237], solves problem 76 of G. Birkhoff's "Lattice theory" [rev. ed., Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948; these Rev. **10**, 673]. Another consequence is that an l-group need not be a Hausdorff space in its interval topology, and need not be a topological group in the usual sense. This answers problem 104. A necessary and sufficient condition is found for an element of a lattice to be isolated in the interval topology. The result provides a partial solution of problem 21 of "Lattice theory". *O. Frink.*

Floyd, E. E., and Fort, M. K., Jr. A characterization theorem for monotone mappings. Proc. Amer. Math. Soc. **4**, 828–830 (1953).

This paper makes use of previous results of E. E. Floyd [Duke Math. J. **16**, 225–235 (1949); these Rev. **10**, 726], M. K. Fort, Jr. [Proc. Amer. Math. Soc. **1**, 59–62 (1950); these Rev. **11**, 381], and J. W. T. Youngs [Ann. of Math. (2) **45**, 753–785 (1944); these Rev. **6**, 278] to obtain the result that a mapping (i.e., continuous function) f on the 2-sphere S onto itself is monotone if and only if f has a continuous extension g which maps the interior of S homeomorphically upon itself. *D. W. Hall.*

Suvorov, G. D. Prime ends of a sequence of plane regions converging to a nucleus. Mat. Sbornik N.S. **33(75)**, 73–100 (1953). (Russian)

The author shows that the whole theory of prime ends developed by Carathéodory for a single simply-connected plane domain [Math. Ann. **73**, 323–370 (1913)] fits in even better, and with almost no change of wording, when this single domain is replaced by a notion which Carathéodory had himself previously introduced [ibid. **72**, 107–144 (1912)], namely that of a sequence of domains converging to a nucleus. In terms of the new prime ends, the author is able to restate many of Carathéodory's results in a more general setting and he observes that there are similar extensions of recent work of J. Lelong-Ferrand [J. Math. Pures Appl. (9) **31**, 103–126, 245–252 (1952); these Rev. **14**, 36]. *L. C. Young* (Madison, Wis.).

Yang, Chung-Tao. On Borsuk's problem. Summa Brasil. Math. **3**, 13–20 (1952).

Let X and Y be topological spaces such that each is homeomorphic to a retract of the other. Borsuk asked [Eilenberg, Ann. of Math. (2) **50**, 247–260 (1949), p. 248; these Rev. **10**, 726] whether their cohomology groups are necessarily isomorphic. The author shows that the answer is affirmative if, for instance, the cohomology groups involved have bases. The proofs depend on algebraic lemmas that the author says he will prove in another paper. *R. H. Fox.*

Hukuhara, Masuo. A family of continuous functions and a mapping. Mem. Fac. Sci. Kyūsyū Univ. A. **5**, 61–63 (1950). (Japanese)

Using a theorem of Tychonoff, the author proves the following theorem which has applications in the theory of functional equations. Let E be a separable metric space and F a convex family of continuous functions on E, closed under the topology of uniform convergence. Let, furthermore, T be a mapping of F into itself such that T is continuous on F with respect to the same topology as above and that the image of F by T is an equi-continuous family of functions on E, bounded at each point of E. Then, F contains a function f which is mapped into itself by the mapping T. *K. Iwasawa* (Cambridge, Mass.).

Calabi, Eugenio, and Eckmann, Beno. A class of compact, complex manifolds which are not algebraic. Ann. of Math. (2) **58**, 494–500 (1953).

This paper defines on the topological product $S^{2p+1}\times S^{2q+1}$ of two spheres of dimensions $2p+1$ and $2q+1$ respectively, $p>0$, a complex analytic structure. The complex manifold so obtained, to be denoted by $M_{p,q}$, has various properties which help to clarify our understanding of complex manifolds in relation to complex algebraic varieties. In fact, $M_{p,q}$ does not admit any Kähler metric, and therefore cannot be analytically imbedded in a complex projective space, and is not algebraic. It admits a complex analytic fibering, with two-dimensional tori as fibers and having as base space the product $P_p\times P_q$ of complex projective spaces of (complex) dimensions p and q respectively. These tori are compact complex analytic sub-manifolds which carry cycles homologous to zero. Moreover, every compact complex sub-manifold of $M_{p,q}$ is the set of all points that are mapped onto an algebraic sub-variety of $P_p\times P_q$ by the fiber projection described above. Finally, let a and a' be points of S^{2p+1} and S^{2q+1} respectively and denote by $E_{p,q}$ the open region in $M_{p,q}$ corresponding to $(S^{2p+1}-\{a\})\times(S^{2q+1}-\{a'\})$. Then $E_{p,q}$ is an open $(p+q+1)$-dimensional complex manifold homeomorphic to a cell and has the property that, for $q>0$, it neither admits a Kähler metric, nor can be covered by a single complex coordinate system, nor contains any non-constant, entire holomorphic function. The construction of the analytic complex structure on $S^{2p+1}\times S^{2q+1}$ is different from that of Hopf's classical example of $S^{2p+1}\times S^1$. *S. Chern* (Chicago, Ill.).

Adem, José. The iteration of the Steenrod squares in algebraic topology. Bol. Soc. Mat. Mexicana **10**, 1–11 (1953). (Spanish)

This is a Spanish translation of a previously published paper [cf. Proc. Nat. Acad. Sci. U. S. A. **38**, 720–726 (1952); these Rev. **14**, 306]. *W. S. Massey* (Providence, R. I.).

Borel, A. La cohomologie mod 2 de certains espaces homogènes. Comment. Math. Helv. **27**, 165–197 (1953).

This paper studies, by means of the spectral algebra, the cohomology mod 2 of fiber bundles with the orthogonal group as structural group. G being a compact Lie group, a universal bundle E_G in the dimension k is a compact connected principal fiber bundle whose homotopy groups of dimension $\leq k$ are zero. Its base space B_G is called a classifying space for G (and for k). Any principal bundle with a compact base space B can be induced by a mapping $B\to B_G$, defined up to a homotopy. This induces a characteristic homomorphism $\sigma^*: H(B_G, \Gamma)\to H(B, \Gamma)$, where Γ is a coefficient ring and $H(B_G, \Gamma)$, $H(B, \Gamma)$ are the cohomology rings

of B_G, B relative to Γ. The image of σ^* is the characteristic subalgebra. Let U be a closed subgroup of G. A universal bundle for G is universal for U, so that there is a mapping $\rho(U, G): B_U \to B_G$, which induces a homomorphism $\rho^*(U, G)$: $H(B_G, \Gamma) \to H(B_U, \Gamma)$. These homomorphisms, and in particular the latter, play an important rôle in the method in this paper.

Let $O(n)$ be the orthogonal group in n variables, $SO(n)$ the special orthogonal group in n variables, $Q(n)$ the subgroup of diagonal matrices of $O(n)$, $SQ(n)$ the subgroup of diagonal matrices of $SO(n)$, and $U(n)$ the unitary group in n complex variables. The coefficient ring will always be the field mod 2. Then $H(B_{Q(n)})$ is isomorphic to the algebra of polynomials in n generators of degree 1 and $H(B_{SQ(n)})$ to the algebra of polynomials in $n-1$ generators of degree one. $H(O(n)/Q(n)) \cong H(SO(n)/SQ(n))$ is generated by its elements of degree $\leqq 1$, and $H(SO(n)/SQ(n))$ is identical with its characteristic subalgebra. The homomorphism

$$\rho(Q(n), O(n)): H(B_{O(n)}) \to H(B_{Q(n)}) \cong Z_2[x_1, \cdots, x_n]$$

is an isomorphism into, its image being the algebra of symmetric functions in $x_1, \cdots, x_n$. The originals of the elementary symmetric functions in $x_1, \cdots, x_n$ under this homomorphism are the Stiefel-Whitney characteristic classes reduced mod 2. This fact reduces the proofs of the Whitney duality theorem mod 2 and formulas of Wu Wen-Tsun on the Steenrod squares of the Stiefel-Whitney classes to the establishment of certain identities on symmetric functions, which the author carried out. The homomorphism $\rho(SO(n), O(n)): H(B_{O(n)}) \to H(B_{SO(n)})$ is onto; its kernel is an ideal generated by a one-dimensional cohomology class, which is the characteristic class mod 2 of the covering $B_{SO(n)} \to B_{O(n)}$.

The method is then applied to derive homology properties of the following homogeneous spaces, and in particular their Poincaré polynomials mod 2:

$$O(n)/O(n_1) \times \cdots \times O(n_k), \quad n = n_1 + \cdots + n_k,$$
$$U(n)/Q(n), \quad U(n)/O(n), \quad G_2/Q(3), \quad G_2/SO(4),$$

where G_2 is the exceptional group in 14 parameters.

S. Chern (Chicago, Ill.).

*Aleksandrov, P. S. **Duality laws and dimension.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 329–357. Akadémiai Kiadó, Budapest, 1952. (Russian. Hungarian summary)

This is an account of problems originating in the field of the Pontryagin duality theorem and homological dimension-theory of the author. §1 sketches the ideas and some of the details of the duality theorem as developed in the author's work. Attention is drawn to work of Smirnov [see these Rev. 13, 268]. §2 on duality for non-closed subsets of S^n reports on work by the author [these Rev. 9, 52 and particularly 9, 456, 457], related theorems of Čogosvili, and mapping-theorems of Dowker [these Rev. 8, 594]. §3 on duality for closed A and complement $R-A$ in a locally bicompact regular space R, discusses the duality theorems of the type of Alexander-Kolmogoroff. §4 on dimension and duality for compact F in euclidean n-space R^n, using finite integral-coefficient chains in $R^n - F$ is concerned with local-obstacle theory, as initiated by the author in his classical paper on "Dimensionstheorie" [Math. Ann. 106, 161–238 (1932)]. Principal emphasis here is on work of Sitnikov [these Rev. 11, 45, 676].

§5 is on "algebraic" problems in dimension-theory associated with homology theory of dimension according to different coefficient groups. Here principal topics are dimension of topological product as worked out by Bokstein [reported in these Rev. 10, 316], the example of Boltyanskiĭ [these Rev. 11, 45] and his work on "full-valuedness" of compacta [these Rev. 11, 195] which defines those compact sets that can figure in topological products whose Urysohn-Menger dimension is always the sum of the separate dimensions of the factors.

The final miscellaneous §6 discusses the problem of dimension-raising maps of special types: e.g., the problem of the existence of an open map of a cube of one dimension on a cube of higher dimension or on a compact set of higher dimension. A footnote added in proof announces that L. Keldyš [cf. these Rev. 14, 71] has just constructed a monotone map of a 3-cube on a 4-cube. In this section, the author reviews results in dimension theory of non-separable spaces, calling especial attention to examples by Lunc [these Rev. 11, 46] and Lokucievskiĭ [these Rev. 11, 46] dealing with possible disparity between covering-dimension and inductive-dimension. He raises the question whether these may not be necessarily equal for non-separable metric spaces.

All of the problems mentioned in this final section of the paper dealing with dimension-raising mappings have been announced as settled in the affirmative, and in essentially complete generality, by R. D. Anderson [see Bull. Amer. Math. Soc. 58, 393, 465–466, 661 (1952); 59, 247–248 (1953); and Trans. Amer. Math. Soc. 73, 211–222 (1952); these Rev. 14, 305]. *L. Zippin* (Flushing, N. Y.).

Ringel, Gerhard. Bestimmung der Maximalzahl der Nachbargebiete auf nichtorientierbaren Flächen. Arch. Math. 4, 137–142 (1953).

This is a summary of a paper by the author which is to appear in Math. Ann. It is concerned with the colouring of maps on one-sided surfaces, and is a continuation of an earlier paper by the author (J. Reine Angew. Math. 190, 129–147 (1952); these Rev. 15, 143). Heawood proved that, for $h \geqq 2$, the number of colours which are necessary to colour the countries of any map drawn on a surface of connectivity h in such a way that any two countries which have a common frontier receive different colours is at most H_h, where $H_h = [3\frac{1}{2} + \frac{1}{2}(24h-23)^{1/2}]$. The author will prove in his forthcoming paper that Heawood's upper bound is always best possible for one-sided surfaces, except in the case $h=3$. He will do this by constructing, for all $h \neq 3$, one-sided surfaces of connectivity h with maps containing H_h mutually adjacent countries. *G. A. Dirac.*

GEOMETRY

Bunt, L. N. H. Equivalent forms of the parallel axiom. Euclides, Groningen 28, 249–267 (1953). (Dutch)

Williams, W. L. A pentagon theorem. Amer. Math. Monthly 60, 616–617 (1953).

Court, N. A. Pascal's theorem in space. Duke Math. J. 20, 417–421 (1953).

The four triads of points in which the four triads of coterminal edges of a tetrahedron T are cut by four transversal planes, respectively, lie on the same quadric, if and

only if, the four secant planes meet the faces of T opposite the respective vertices in four lines which are either coplanar or hyperbolic. *O. Bottema* (Delft).

Marmion, A. Sur les six points de Servais d'un tétraèdre. Mathesis 62, 215–233 (1953).

Coxeter, H. S. M. The golden section, phyllotaxis, and Wythoff's game. Scripta Math. 19, 135–143 (1953).

Rangaswami Aiyer, K. On certain cubic transformations in circle space associated with pedal and contact circle systems. III. Geometry of pedal and contact circle systems. J. Annamalai Univ. 18, 163–172 (1953).

Deaux, R. Cubiques anallagmatiques. Mathesis 62, 193–204 (1953).

Droussent, Lucien. Cubiques circulaires anallagmatiques par points réciproques ou isogonaux. Mathesis 62, 204–215 (1 plate) (1953).

***Struik, Dirk J. Lectures on analytic and projective geometry.** Addison-Wesley Publishing Co., Cambridge, Mass., 1953. ix+291 pp. $6.50.

In der derzeitigen mathematischen Ausbildung der Studenten kommt nach Ansicht des Verf. die moderne Geometrie des 19. Jahrhunderts vielfach zu kurz, worunter das tiefere Verständnis anderer mathematischer Disciplinen leidet. Diesem Mangel durch einen einsemestrigen Kursus absuhelfen, setzt sich das vorliegende Buch zum Ziel. Der Leser soll den üblichen dreisemestrigen Kursus über analytische Geometrie und Calculus absolviert haben. Das Buch behandelt die klassische projektive Geometrie in 1, 2 und 3 Dimensionen algebraisch, gelegentlich wird auch von der synthetischen Darstellung Gebrauch gemacht. Nachbargebiete der Geometrie, wie Algebra, Vektoranalysis und Statik werden berührt. Der Leser lernt die topologischen Modelle der projektiven Geometrie und die Anfangsgründe der darstellenden Geometrie kennen. Das Buch enthält historische Hinweise und zahlreiche Übungsaufgaben. Der Stoff ist übersichtlich gegliedert und die Darstellung leicht fasslich, sie wird durch gute Figuren ergänzt. Überschriften der 11 Kapitel: Point sets on a line; Line pencils; Line coordinates. Homogeneous coordinates; Transformations of the plane; Projective theory of conics; Affine and Euclidean theory of conics; Projective metric; Points, lines and planes; Projective theory of quadrics; Affine and Euclidean theory of quadrics; Transformation of space. *R. Moufang.*

Morgantini, Edmondo. Sulla configurazione di quattro rette di uno spazio proiettivo a quattro dimensioni. Atti Accad. Ligure 9 (1952), 29–35 (1953).

The author shows how four lines of general position in projective 4-space determine a fifth "associated" line, such that every plane meeting any four of the five lines meets the fifth also [cf. H. F. Baker, Principles of geometry, vol. 4, Cambridge, 1925, pp. 113–124]. *H. S. M. Coxeter.*

Di Noi, Salvatore. Interpretazione cinematica d'una geometria due volte parabolica nel piano. Archimede 5, 145–155 (1953).

***Skopec, Z. A. Cyclographic mapping of Lobačevskiĭ space.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 129–150. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

The space of Lobačevskiĭ is divided by a plane π, the image plane, into two separate regions. To every plane σ belongs another plane σ' symmetrical with respect to π. The plane σ can be divergent from π, parallel to π, or intersect π. When σ is divergent, then it is possible to associate to it in a one-to-one way an oriented cycle of π; the radius r of this cycle is related to the shortest distance of π and σ by the equation $\Pi(h)+\Pi(r)=\pi/2$, the center of the cycle is at the foot of the common perpendicular. The same circumference with opposite sense serves as an image to σ'. When σ is parallel to π the one-to-one correspondence is performed by means of an oriented horicycle obtained by intersecting π with the equidistant surface to σ of altitude h where $\Pi(h)=\pi/4$. In the case that σ intersects π in a line b at acute angle φ the image of σ consists of the oriented equidistant curves to b at distance m, where $\Pi(m)=\varphi$. These curves consist of one positively and one negatively oriented branch. When $\varphi\to\pi/2$, then $m\to 0$ and the image is the line b with double orientation. This correspondence is called the cyclometric mapping of the space on the plane π. Under it a line, considered as the support of a plane pencil, passes into the common chord of a pencil of equally oriented cycles. Two perpendicular lines in a plane pass through each other's pole with respect to the cycle-image of the plane. Of the many other properties of the cyclometric mapping we mention that if two intersecting planes are mapped into two cycles, their angle is equal to the angle of the cycles. *D. J. Struik* (Cambridge, Mass.).

Tresse, A. Théorie élémentaire des géométries non euclidiennes. Bull. Soc. Math. France 81, 81–143 (1953).

The paper is of an expository nature. The best known properties of plane non-euclidean geometry are derived by means of inversion. *H. Busemann* (Los Angeles, Calif.).

Sholander, Marlow. Plane geometries from convex plates. Pacific J. Math. 3, 667–671 (1953).

Let K be a two-dimensional convex domain. On the chords of K parallel to the direction φ let us consider the points P which divide the chord in the ratio i to $1-i$. The locus of P will be an open Jordan arc $s_i(\varphi)$ called a "strut". Two struts with no common points or all points in common are called parallel; for instance, $s_i(\varphi)$, $s_k(\varphi)$ are parallel. If I represents the set of inner points of K and K has at most one edge and has no corners, the author proves the following properties: (a) two distinct points in I lie on one and only one strut; (b) given a strut s and a point P in I, there is one and only one strut through P and parallel to s; (c) there are three points of I not on a strut. These properties are Axioms I, II and III of an affine geometry in the sense of Artin [Rep. Math. Colloquium (2) 2, 15–20 (1940); these Rev. 3, 179]. Consequently, to each K which satisfies the mentioned conditions is associated an affine geometry and, by adjoining the boundary of K as ideal line, also a projective geometry. *L. A. Santaló* (Buenos Aires).

Blumenthal, Leonard M. Boolean geometry. I. Rend. Circ. Mat. Palermo (2) **1** (1952), 343–360 (1953).

It is known that in a Boolean algebra the symmetric difference $d(a, b)=ab'+a'b$ has the formal properties of a metric, thus giving rise to corresponding definitions of betweenness, congruence and motion. The author studies two notions of linearity for m-tuples of elements. He also defines a segment S_a^b as a set congruent to a maximal chain Γ_α^β between elements α and β; the elements a and b corresponding to α and β are called the endpoints of S_a^b. A typical theorem states that if S_a^b and S_b^c are segments with endpoints a, b and b, c respectively, and if b is between a and c, then $S_a^b \cup S_b^c$ is a segment. *B. Jónsson.*

Convex Domains, Extremal Problems, Integral Geometry

***Vincze, István. Über die Schwerlinie einer geschlossenen, konvexen Kurve.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 679–687. Akadémiai Kiadó, Budapest, 1952. (Hungarian. Russian and German summaries)

The midpoints of the chords of a plane closed convex curve, containing no straight segments, which are parallel to a given direction, form the load line of the convex curve corresponding to this direction. The midpoints of those chords of such a curve which pass through a point P in the plane of the curve form the load line of the curve corresponding to P. Inequalities connected with these curves are derived. In the first part of the paper only plane convex curves containing no straight segments are considered, in the last part the results derived are extended, as far as possible, to plane convex curves which may contain straight segments.

The starting point of the paper is the theorem that the distance between the midpoints of two opposite sides of a plane quadrilateral is less than or equal to the arithmetic mean of the lengths of the other two sides. Equality holds for a convex trapezium only. The following inequalities are established.

I. Let C be a closed convex plane curve containing no straight segment, and let L be the circumference and D be the diameter of C. Let $l(\phi)$ denote the length of the load line corresponding to those chords of C which make an angle ϕ, measured in a given sense, with some fixed direction in the plane. Then $D \leqq \max_\phi l(\phi) < L/2$.

II. Let $l(P)$ be the length of the load line of C corresponding to the point P. Then $l(P) \leqq L/2$. Equality holds if and only if P lies on C.

III. If P lies inside or on C, the load line of P will be a closed curve or a point. In this case let $t(P)$ denote the area enclosed by it. Let T denote the area enclosed by C. Then $t(P) \leqq T/4$. Equality holds if and only if P lies on C.

IV. Let d denote the thickness of the ring of minimal thickness which contains C. Then $\min_P t(P) \leqq \frac{1}{4}\pi d^2$. Here $\min t(P)$ represents the asymmetry of the curve in a sense.

If C may contain straight segments, the inequalities I and II still hold, but the load lines are no longer unique. However, with suitable, fairly obvious, definitions, which remove the ambiguities, inequality III still holds. The author puts forward the following two problems. (a) How can the inequalities be sharpened, possibly placing restrictions on C? (b) For which convex curves of given area is $\min t(P)$ greatest? *G. A. Dirac* (London).

Pogorelov, A. V. On existence of a convex surface with a given sum of the principal radii of curvature. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 127–130 (1953). (Russian)

The sum $F(u)$ of the principal radii of curvature of a convex surface satisfies as function of the unit normal u the conditions $F(u) \geqq 0$, $\int_\Omega u F(u) d\omega = 0$, where Ω is the unit sphere. It is known that not every function $F(u)$ satisfying these conditions is the sum of the principal radii of curvature of a suitable convex surface. It is shown that it suffices to add the condition that for every u and every direction t through u the relation $F(u) - F_{tt}(u) \geqq 0$, where $F_{tt}(u)$ is the second derivative of $F(u)$ with respect to arclength along the great circle with direction t. *H. Busemann.*

Pogorelov, A. V. On stability under bending of isolated edge points on a convex surface. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 131–134 (1953). (Russian)

A surprising and nontrivial example is constructed by using some of the deeper results of A. D. Alexandrov to show that isolated points of a convex surface at which the number of independent supporting planes equals two may arise and disappear under isometric deformation of the surface. *H. Busemann* (Los Angeles, Calif.).

Kirsch, A. Die Pferchkugel eines Punkthaufens. Ein elementargeometrischer Beweis für den Satz von Jung im räumlichen Falle. Math.-Phys. Semesterber. **3**, 214–217 (1953).

The note offers a proof of Jung's theorem (concerning the smallest covering sphere) for finite subsets of E_3. No mention is made of a paper by Blumenthal and Wahlin [Bull. Amer. Math. Soc. **47**, 771–777 (1941); these Rev. **3**, 90] in which the general theorem (arbitrary subsets of E_n) is proved very simply. *L. M. Blumenthal.*

Taylor, S. J. Some simple geometrical extremal problems. Math. Gaz. **37**, 188–198 (1953).

The problems this paper deals with are as follows: To find among the class of convex curves of given perimeter and either lying in a given circle or lying in a given Reuleaux triangle or having a given diameter that one which encloses the minimum area. The first two problems are completely solved while in the direction of the third one certain partial results are obtained. In the first case the extremal curve is an inscribed $(n+1)$-gon $(n \geqq 1)$ of n equal sides of length, say, b and a side of length $\leqq b$. The author points out that this result was previously obtained in a somewhat different way by Favard [Ann. Sci. Ecole Norm. Sup. (3) **46**, 345–369 (1929)]. *L. Fejes Tóth* (Budapest).

Zalgaller, V. A. On a necessary condition for the densest distribution of figures. Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 153–162 (1953). (Russian)

Necessary conditions for distributing given plane figures with sufficiently smooth boundaries in a given polygonal domain with a minimum amount of waste are established as follows. The figures are considered as two-dimensional rigid bodies which touch each other without friction. If they are subject to external forces (acting in the plane), then $dW = -dS$, where W is work and S is area. The necessary condition $dS \geqq 0$ then becomes $dW \leqq 0$. The author shows with a variety of examples how this simple principle leads to concrete results in special cases. *H. Busemann.*

Eggleston, H. G. On Rado's extension of Crum's problem. J. London Math. Soc. **28**, 467–471 (1953).

Starting from a problem of Crum solved by Besicovitch, R. Rado [same J. **22**, 287–289 (1948); these Rev. **9**, 605] showed that to any integer k satisfying $1 \leqq k \leqq \frac{1}{2}(n+1)$ there can be found an infinite sequence of non-overlapping n-dimensional polytopes of n-space having the property that any k of the sequence have a $(n-k+1)$-dimensional intersection. The author shows that in case of $k > \frac{1}{2}(n+1)$ this assertion does not hold any more. What is the maximum number N of polytopes in case of $\frac{1}{2}(n+1) < k \leqq n$ having the above intersection property? The result of Rado $N \geqq n+2$ concerning this question is completed here by showing that $N = n+2$. *L. Fejes Tóth* (Budapest).

*Fejes Tóth, L. **Lagerungen in der Ebene, auf der Kugel und im Raum.** Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LXV. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. x+197 pp. DM 24.00; bound DM 27.00.

This book is a collection of recent work done by Hadwiger, van der Waerden, the author, and many others, on problems of densest packing and thinnest covering in the plane, on the sphere, and in space, and numerous other extremal problems. The regions considered are simple and connected, in two or three dimensions.

Chapter I contains proofs of geometrical theorems which are used in the subsequent work and are also of interest in themselves. Direct and indirect proofs are given of the following theorem: Let r and R be the inradius and circumradius of a convex n-gon of area F and perimeter L. Then

$$nr^2 \tan\frac{\pi}{n} \leqq F \leqq \tfrac{1}{2}nR^2 \sin\frac{2\pi}{n}, \quad 2nr \tan\frac{\pi}{n} \leqq L \leqq 2nR \sin\frac{\pi}{n},$$

with equality in all four cases only if the polygon is regular. Thus, among all convex n-gons inscribed in (circumscribed about) a given circle, the regular n-gon has maximum (minimum) area and perimeter. Several interesting inequalities concerning triangles are proved. For example, if R_1, R_2, R_3 are the distances from an arbitrary point O to the vertices of a triangle of area Δ, then

$$R_1 + R_2 + R_3 \geqq 2 \cdot 3^{1/4} \Delta^{1/2}$$

with equality only if the triangle is equilateral and O is its centre. A short treatment of the regular and Archimedean polyhedra (which includes a proof of Euler's formula) is accompanied by clear drawings of these bodies.

Chapter II deals with the problem of approximating convex regions by inscribed and circumscribed polygons. Let T be (the area of) a convex region, and T_n the inscribed n-gon of maximum area. It is shown that

$$T_n \geqq T \frac{n}{2\pi} \sin\frac{2\pi}{n},$$

with equality only if T is an ellipse. Thus the approximation of the area of a convex region by inscribed polygons is worst when the region is an ellipse. The corresponding problem for circumscribed polygons has not been solved. Nevertheless, for large n the external curve is shown to approximate a circle.

The similar problem of approximating the boundary of a convex region by in-(circum-)scribed polygons is unsolved, but the following inequality is proved. For any convex region there are circumscribed and inscribed n-gons of perimeter L_n and l_n such that

$$\frac{L_n - l_n}{L_n} \leqq 2 \sin^2\frac{\pi}{2n}.$$

Chapter II also contains an exposition of Blaschke's concept of the affine length of a curve.

Let a countable number of unit circles be packed (i.e., placed without overlapping) in the plane. Let $N(R)$ be the number of these circles wholly contained in a large circle of radius R and centre O. The density d of the packing is defined to be

$$d = \lim_{R\to\infty} \frac{N(R)\pi}{\pi R^2} = \lim_{R\to\infty} \frac{N(R)}{R^2}.$$

If the circles are placed so as to cover the plane (i.e., so that every point of the plane is contained inside, or on the boundary of, at least one circle), the above limit defines the density D of the covering. Chapter III contains proofs that

$$d \leqq \pi/\sqrt{12} = .9069\cdots, \quad D \geqq 2\pi/\sqrt{27} = 1.209\cdots.$$

In both cases equality is attained only when the centres of the spheres are the points of a lattice formed by equilateral triangles.

The problem of how best to cover a finite convex region is considered, and it is proved that if T is (the area of) a convex region of boundary L, then

$$\left[\frac{2}{3\sqrt{3}}T + \frac{2}{\pi\sqrt{3}}L + 1\right]$$

unit circles suffice to cover T. A similar formula for the maximum number of unit circles which can be packed into a convex region is derived.

Inequalities are established for bounds of the density of a packing of circles of n different radii. Also considered is the more general problem of packing congruent convex regions into a larger convex region. The following inequality seems particularly interesting. If n congruent convex regions G are packed into a convex hexagon of area S, then $n \leqq S/s$, where s is the area of the smallest hexagon circumscribed about G.

Let P be (the area of) the fundamental parallelogram of a lattice generated by a translation group. If the regions obtained by applying the translations to a region (of area) G do not overlap, then the density of this lattice packing is G/P. If the regions arranged in this way cover the plane, the above ratio gives the density of the lattice covering. Let $d(G)$ and $D(G)$ be the densities of the densest lattice packing of G and the thinnest lattice covering by G. Chapter IV considers the problem of finding bounds for these densities. If G is convex, it is shown that

$$d(G) \geqq 2/3, \quad D(G) \leqq 3/2,$$

with equality in both cases only when G is a triangle. If G is centrally symmetrical as well as convex the situation is understandably different. In this case it is shown that $D(G) \leqq 2\pi/\sqrt{27}$, with equality only if G is an ellipse. The smallest value for $d(G)$ has not been established. For an ellipse it is $\pi/2\sqrt{3} = .9069\cdots$, but for a regular octagon it is $4(3-\sqrt{2})/7 = .90616\cdots$ and for a "smoothed" octagon (diagram on page 104) it is

$$(9 - 4\sqrt{2} - \log 2)/(2\sqrt{2} - 1) = .9024\cdots.$$

The author agrees with Mahler's conjecture that this last density is the extreme one.

Chapter V contains numerous theorems which exhibit the extremal properties of regular and semi-regular polyhedra. For example, the following three theorems are proved. (i) If $n \geqq 3$ congruent spherical caps are packed on the surface of a sphere, and if d is the density of the packing, then

$$d \leqq \frac{n}{2}(1-\tfrac{1}{2}\operatorname{cosec} w_n), \quad \text{where} \quad w_n = \frac{n}{n-2}\frac{\pi}{6}.$$

(ii) If $n \geqq 3$ congruent spherical caps cover the surface of a sphere, and D is the density of the covering, then

$$D \geqq \frac{n}{2}(1-3^{-1/2}\cot w_n).$$

In (i) and (ii) equality occurs only when $n=3$, 4, 6, 12 and the centres of the caps are the vertices of an equilateral triangle (inscribed in a great circle), a regular tetrahedron, octahedron, or icosahedron. (iii) Let V be the volume, and e, f, and k the number of vertices, faces, and edges of a polyhedron which contains the unit sphere. Then

$$V \geqq \frac{k}{3}\sin\frac{\pi f}{k}\left(\tan^2\frac{\pi f}{2k}\tan^2\frac{\pi e}{2k}-1\right),$$

with equality only if the polyhedron is regular and circumscribed about the sphere.

The isoperimetric theorem is discussed and its solution is given in the form (due to M. Goldberg): If a convex polyhedron of n faces has volume V and surface F, then

$$\frac{F^3}{V^2} \geqq 54(n-2)\tan w_n(4\sin^2 w_n - 1).$$

How should n points be placed on the surface of a unit sphere so that the minimum distance between the $\frac{1}{2}n(n+1)$ pairs of points is maximum? Theorem (i) above solves this problem for $n=3$, 4, 6, 12. Chapter VI contains the recent work of van der Waerden, Habricht and Schütte for other values of n. For $n=5$, 7, 8, 9 the problem has been solved. The configurations are more or less irregular, and the methods used for a particular n do not necessarily help in solving the problem for other values of n. For $n=10$, 11, 13, 14, 15, 16, 32, the author gives Schütte and van der Waerden's conjectures for the best configurations.

Chapter VII contains a discussion of the problems of the densest packing of equal spheres, and the thinnest covering of space by equal spheres. If the condition is imposed that the centres of the spheres constitute a lattice, the problems have been solved: The best packing density is $\pi/\sqrt{18}=.74048\cdots$ and is attained when the centres of the spheres are the points of a face-centered cubic lattice, and the thinnest covering density is $5^{3/2}\pi/24=1.464\cdots$ which is attained when the centres form a body-centred cubic lattice.

Although it is widely believed that $\pi/\sqrt{18}$ is the greatest density without the lattice restriction, this has never been proved. Let d be the density of an arbitrary packing of spheres. Blichfeldt [Math. Ann. 101, 605–608 (1929)] showed $d<.835$ and Rankin [Ann. of Math. (2) 48, 1062–1081 (1947); these Rev. 9, 226] improved this to $d<.828$. The author gives a proof (different from Blichfeldt's) that $d<.835$, and a non-rigorous proof that $d<.7545$, but observes that great technical difficulties must be overcome before the conjecture $d \leqq \pi/\sqrt{18}$ can be proved.

Each chapter begins with a description of the problems to be discussed and ends with a historical outline, in which the author gives the sources of the theorems and proofs. There is an excellent bibliography (which includes works as recent as 1952). The author indicates the direction of further research. The exceptionally clear treatment of the variety of problems, and the 124 beautiful diagrams, combine to make this book extremely attractive. *W. O. Moser.*

Algebraic Geometry

Marchionna, Ermanno. Sulle quartiche piane razionali invarianti per un gruppo trirettangolo di omografie. Period. Mat. (4) 31, 229–245 (1953).

Guillotin, M. R. Courbes associées à un quadrilatère. Bull. Soc. Roy. Sci. Liège 22, 327–340 (1953).

Lorent, H. Courbes engendrées par deux ensembles de lignes planes à deux paramètres. I, II. Bull. Soc. Roy. Sci. Liège 22, 317–326, 388–395 (1953).

Godeaux, Lucien. Ancora sopra una particolare involuzione di Geiser. Rend. Sem. Fac. Sci. Univ. Cagliari 22 (1952), 1–2 (1953).

Si mostra che une critica del Prof. Turri ad un risultato stabilito in una nota dello stesso titolo [stessi Rend. 21, 1–3 (1952); questi Rev. 14, 77] è senza valore.

Sunto del autore.

Turri, Tullio. Superficie razionali reali di S_3 rappresentabili su un piano. Rend. Sem. Fac. Sci. Univ. Cagliari 22 (1952), 3–15 (1953).

Turri, Tullio. Sulla classificazione delle trasformazioni involutorie del piano. Rend. Sem. Fac. Sci. Univ. Cagliari 22 (1952), 16–26 (1953).

Pisano, Paolo. Rappresentazione piana di involuzioni sopra superfici cubiche. Rend. Sem. Fac. Sci. Univ. Cagliari 22 (1952), 27–33 (1953).

***Piazzolla-Beloch, Margherita. Proprietà diametrali delle superficie algebriche.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 425–430. Casa Editrice Perrella, Roma, 1953.

The "diametral theory" developed by the author for plane curves [Teoria diametrale delle curve algebriche piane, Ist. Mat. Univ. Ferrara, 1949=Ann. Univ. Ferrara. Parte I. 7, 39–121 (1948); these Rev. 11, 684; 12, 1002, 736] is extended to surfaces in Euclidean three-dimensional space. The definitions and results are entirely similar to those of the paper cited. In equation (1), p. 425, t^i should be t^{n-i}. It is stated that the number of principal diametral planes (i.e., diametral planes perpendicular to the conjugate directions) is not greater than n^2; the argument given shows in fact that this number is n^2-n+1. *P. Du Val.*

Benedicty, Mario. La geometria algebrica astratta e il concetto di varietà algebrica. Archimede 5, 133–144 (1953).

Expository paper.

Segre, Beniamino. Sul massimo numero di nodi delle superficie algebriche. Atti Accad. Ligure 9 (1952), 15–22 (1953).

This is an attempt to find out something about the function $\mu(n)$ defined as the maximum number of isolated

double points on an algebraic surface of order n. It is well known that $\mu(1)=0$, $\mu(2)=1$, $\mu(3)=4$, $\mu(4)=16$. Bassett [Nature 73, 246 (1906); Quart. J. Pure Appl. Math. 38, 63–83 (1906)] shewed that

$$\mu(n)\leqq\tfrac{1}{2}\{n(n-1)^2-5-[n(n-1)3n-14)+25]^{\frac{1}{2}}\}.$$

The present paper uses the transformation

$$x_0':x_1':x_2':x_3'=x_0^2:x_1^2:x_2^2:x_3^2 \tag{1}$$

to construct surfaces of order $2n$ with a large number of double points, given one of order n with the maximum number; from which in any case it follows that $\mu(2n)>8\mu(n)$.

The author gives without proof the theorem that the maximum number of double points on an algebraic surface which (a) has not an infinity of tritangent planes, (b) has a parabolic and a flecnodal curve with no common part, is

$$d_n\leqq\tfrac{1}{12}[7n^3-18n^2+11n-54 - (n^6-12n^5+46n^4+336n^3-2639n^2+3420n+2916)^{\frac{1}{2}}].$$

On the assumptions that $d_n=\mu(n)$ this gives $\mu(5)\leqq 31$, $\mu(6)\leqq 63$. As the author constructs by (1) a sextic surface with 63 nodes, and Togliatti [Vierteljschr. Naturforsch. Ges. Zürich 85 Beiblatt, 127–132 (1940); these Rev. 3, 15] found a quintic with 31 nodes, it seems probable that $\mu(5)=31$, $\mu(6)=63$.

Nothing is said about $\mu(7)$. Again applying (1) to a Kummer surface, the author obtains an octavic surface with 153 nodes, and this number is raised to 160 if it is possible to have a Kummer surface both inscribed and circumscribed to the tetrahedron of reference and also touching its edges, which a crude count of constants renders plausible. The best upper bound for $\mu(8)$ however is 175.

Bassett's formula, together with $\mu(2n)>8\mu(n)$, shews that if $\mu(n)/n^k$ tends to a limit k as n tends to infinity, we must have $k=3$, and $\mu(n)/n^3<k<\frac{1}{2}$ for all n. On this assumption we find the best lower bound for k is that given by $\mu(8)\geqq 153$, giving $k>153/512$, or $k>5/16$ if we assume $\mu(8)\geqq 160$.

P. Du Val (Bristol).

Gallarati, Dionisio. Alcune riflessioni intorno ad una nota del prof. B. Segre. Atti Accad. Ligure 9 (1952), 106–112 (1953).

This paper is a brief commentary on one by Segre [see the preceding review]. The transformation $\rho y_i=x_i^p$ $(i=0,\cdots,r)$ is used in the same way as Segre's special case of this for $r=3$, $p=2$. In particular, for $r=4$, $p=2$, we obtain a quartic primal in S_4 with 40 nodes from a general quadric inscribed in the simplex of reference. It is shewn that the tangent cone at each of these has point vertex and does not pass through any of the other nodes; hence the apparent contour of this primal from one of its nodes is a sextic surface with 63 nodes (39 arising from the other nodes of the primal and 24 from lines on it through the node in question). It is not made clear whether this is the same as Segre's sextic with 63 nodes or not.

Finally, by applying the transformation to a surface with the maximum number of nodes for its order, using a tetrahedron of reference in which the surface is inscribed, the relation $\mu(2n)>8\mu(n)$ is amended to $\mu(2n)\geqq 8\mu(n)+16$ whence Segre's lower bound of 153/512 for $\lim_{n\to\infty}\mu(n)/n^3$ is slightly improved to $153/512+1/224$. *P. Du Val.*

Burau, Werner. Projektive Klassifikation der Grassmannrelationen und Kennzeichnung der Minimalmodelle für die Gesamtheiten der verallgemeinerten Raumelemente des S_n. Ann. Mat. Pura Appl. (4) 34, 133–160 (1953).

After some preliminary study of the Grassmannian $G_{n;k}$ of S_k's in S_n (i.e., the locus of points in S_p, where $p=\binom{n+1}{k+1}-1$, whose coordinates are the Plücker coordinate of an S_k in S_n), the author considers the "Inzidenzmannigfaltigkeit" $J_{n;k,h}$ each of whose points corresponds to an S_k lying in an S_h in a fixed S_n; this is constructed as follows: the totality of figures in S_r consisting of an S_h and an S_k is mapped by the points of the Segre product $G_{n;k}\times G_{n;h}$, i.e., the locus of points in S_q, where $q=\binom{n+1}{k+1}\binom{n+1}{h+1}-1$, whose coordinates are all the products of a coordinate of P with a coordinate of Q, where P, Q are arbitrary points of $G_{n;k}$, $G_{n;h}$ respectively; the conditions for S_h to lie in S_k are bilinear in the coordinates of P, Q, and hence linear in those of the corresponding point of the product; thus $J_{n;k,h}$ is a particular linear section of $G_{n;k}\times G_{n;h}$. In particular $J_{n;n-1,0}$ is a prime (hyperplane) section of $S_n\times S_n$. Unfortunately, no general expression is given for the dimensions of the ambient of $J_{n;k,h}$. It is shown however that this ambient is spanned by those of $J_{n-1;k,h}$, $J_{n-1;k,h-1}$, $J_{n-1;k-1,h}$, and $J_{n-1;k-1,h-1}$, from which, together with the facts that $J_{n;n,h}=J_{n;h,-1}=G_{n;h}$, and $J_{n;h,h}$ is the projective model of the quadric sections of $G_{n;h}$, a formula ought to be obtainable by induction.

The main theorem is the following: when the quadrics in the ambient of $G_{n;k}$ $(n\geqq 2k+1)$ are represented by points of S_N, where $N=\frac{1}{2}\left\{\binom{n+1}{k+1}-1\right\}\left\{\binom{n+1}{k+1}+2\right\}$, we have in S_N the following figures, whose points represent quadrics related in special ways to $G_{n;k}$:

$$G_{n;k},\quad J_{n;k+2,k-2},\quad J_{n;k+4,k-4},\quad\cdots,$$

ending with $J_{n;2k+1,-1}=G_{n;k}$ or $J_{n;2k,0}$ according as k is odd or even. The points of $G_{n;k}$ represent the doubled primes in the ambient of the original $G_{n;k}$ which correspond to the most special linear complexes in S_r (of S_k's meeting a fixed S_{n-k-1}), those of each $J_{n;k+2i,k-2i}$ represent quadrics containing $G_{n,k}$, and each related (in a manner too complicated to be explained here) to a particular figure of S_{k-2i} lying in S_{k+2i} in S_r. The ambients of $G_{n;k}$, and of the various $J_{n;k+2i,k-2i}$'s are linearly independent and span S_N. In particular, the join of the ambients of all the $J_{n;k+2i,k-2i}$'s corresponds to the linear system of quadrics of which $G_{n;k}$ is the complete intersection. *P. Du Val.*

Permutti, Rodolfo. Sulla teoria generale dei connessi di dimensione qualunque e sulla base di varietà appartenenti al prodotto di varietà non lineari. Ricerche Mat. 2, 128–139 (1953).

A connex is a subvariety of the Segre variety, which is a projective model of r-way projective space

$$S_{n_1\cdots n_r}=S_{n_1}\times\cdots\times S_{n_r}.$$

The Segre variety arises naturally in any study of algebraic correspondences, and is one of the types of algebraic variety for which a basis (under algebraic equivalence) is known. [For full details see Hodge and Pedoe, Methods of algebraic geometry, vol. II, Cambridge, 1952, these Rev. 13, 972.] The author of the paper under review makes some interest-

ing deductions from the known theorems, obtaining in particular a form of Bézout's theorem for the intersection of two connexes of complementary dimensions. By implication he assumes the existence of a rigorous theory of intersection multiplicities in Segre varieties, but the above reference is not given. Some special connexes are considered. These arise in $S_{n_1} \times S_{n_2}$ if the component variety in S_{n_1} is a rational curve or surface. *D. Pedoe* (Khartoum).

Brusotti, Luigi. Sopra alcune questioni di geometria suggerite dalla teoria delle equazioni a derivate parziali totalmente iperboliche. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 381–404 (1953).

L'auteur considère ici en premier lieu, dans un espace projectif S_p à p dimensions, des hypersurfaces algébriques réelles $\Omega(p, 2n)$, de l'ordre $2n$, dont la partie réelle est homéomorphe au système de n hypersphères concentriques et différentes entre elles; la condition nécessaire et suffisante pour qu'une hypersurface algébrique de l'ordre $2n$ appartienne au type $\Omega(p, 2n)$ s'exprime en disant qu'il existe un point réel O tel que toute droite passant par O rencontre l'hypersurface en $2n$ points réels et tous différents entre eux. Cette condition peut être exprimée facilement sous une forme algébrique. On peut considérer des hypersurfaces telles que $\Omega(p, 2n)$ aussi dans un espace affine. Comme généralisation, on peut ajouter à la partie réelle de $\Omega(p, 2n)$ une $(n+1)$ème composante qui soit homéomorphe à un hyperplan extérieur aux n hypersphères concentriques; on peut encore permettre à l'hypersurface $\Omega(p, 2n)$ de posséder des points multiples convenables. D'un autre côté, on peut imposer à l'hypersurface $\Omega(p, 2n)$ la condition que sa partie réelle se compose de n ovales. Des exemples d'hypersurfaces des types envisagés peuvent être obtenu aisément par la méthode bien connue de la "petite variation" en partout d'hypersurfaces simples; de nombreuses figures, dans le cas $p=2$, montrent l'application de cette méthode. *E. G. Togliatti* (Genova).

Differential Geometry

Franckx, Ed. Sur la théorie des courbes qui appartiennent à une surface et y ont un contact d'ordre k. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39**, 629–635 (1953).

Let $E^{(k)}$ be a set of curves on a surface having contact of order k at a regular point M. Let $\bar{t}$ be their common tangent vector at M, $\bar{n}$ the normal to the surface at M, and $\bar{g}$ the third vector of the trihedral of Darboux. Write the equations

$$\frac{d^k\bar{t}}{ds^k} = a^k_{11}\bar{t} + a^k_{12}\bar{n} + a^k_{13}\bar{g},$$

$$\frac{d^k\bar{n}}{ds^k} = a^k_{21}\bar{t} + a^k_{22}\bar{n} + a^k_{23}\bar{g},$$

$$\frac{d^k\bar{g}}{ds^k} = a^k_{31}\bar{t} + a^k_{32}\bar{n} + a^k_{33}\bar{g},$$

where the scalars a^1_{12} and a^1_{13} are the classical local invariants of Meusnier and Bonnet for a set of curves $E^{(1)}$. The author proves by induction that in going from one curve of a set $E^{(k)}$ to another only a^k_{13} and a^k_{31} of the scalars above will vary, and of these only one will be independent, say a^k_{13}. Moreover, the subset of curves of a set $E^{(k)}$ which correspond to the same parameter value constitute a set $E^{(k+1)}$ and conversely. *A. Schwartz* (New York, N. Y.).

Unger, G. Maximalstetige Kurven. Eine neue Charakterisierung der Kneser-Juelschen Bögen. Elemente der Math. **8**, 79–85 (1953).

Unger, Georg. Ein Kriterium für die Kneser-Juelschen Kurven. Arch. Math. **4**, 143–153 (1953).

Given two differentiable arcs $C: P=P(t)$ and $C': P'=P'(t)$ in the projective plane with the continuous tangents $p(t)$ and $p'(t)$ respectively $(0<t<1)$. A duality is supposed to exist which maps $P(t)$ and $p(t)$ on $p'(t)$ and $P'(t)$ respectively. Let $P(t)$ and $P'(t)$ have the characteristics (a_0, a_1) and (a_0', a_1'). The pairs a_0, a_0' are used to classify the points $P(t)$.

C is "maximally continuous" at $t=t_0$ if both C and C' are strongly differentiable there. The author proves various results which in the standard language of the theory of differentiable curves amount to the equivalence of the following properties of C: (i) C is maximally continuous everywhere, (ii) $a_0=a_0'=1$ for every t, (iii) $a_0=a_1=a_0'=a_1'=1$ for every t, (iv) each point of C has the order 2. He also constructs a C which is nowhere maximally continuous and a point $P(t)$ of characteristic (1, 1) on a strongly differentiable arc C such that $P'(t)$ has the characteristic $(1, \infty)$. *P. Scherk* (Saskatoon, Sask.).

Lane, N. D., and Scherk, Peter. Differentiable points in the conformal plane. Canadian J. Math. **5**, 512–518 (1953).

This paper defines, discusses, and classifies differentiable points on arcs in the conformal plane. Consider the circle through three distinct points, P, Q, R of the conformal plane, at least the first two of which lie on a given arc. If this circle has a limit circle as Q approaches P along the arc, this circle is called a tangent circle to the arc at P. Consider the tangent circle at P which passes through a distinct point R of the arc. If this has a limit circle as R approaches P along the arc, the limit circle is called the osculating circle at P and P is called a differentiable point of the arc. The support and intersection properties of circles through P with respect to the arc are then studied. It is shown, for example, that if P is a differentiable point then every non-osculating circle either supports or intersects at P, and, unless the osculating circle is a point circle, every non-osculating tangent circle supports. In case the osculating circle is a point circle, the other tangent circles all support or they all intersect. A complete classification is given of the possible behavior of the arc with respect to its osculating circle at a differentiable point of the arc. *S. B. Jackson* (College Park, Md.).

Boaga, Giovanni. Risoluzione del triangolo geodetico ellissoidico per mezzo degli elementi della sua rappresentazione conforme sul piano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 349–353 (1953).

Formulas are developed correlating the solution of the geodetic triangle on an ellipsoid with the information supplied by the coordinates of the vertices of the plane conformal representation of the triangle. *N. A. Hall.*

Ščerbakov, R. N. On the generation of surfaces by lines. Uspehi Matem. Nauk (N.S.) **8**, no. 2(54), 147–156 (1953). (Russian)

The general problem considered in this paper may be formulated as follows: On a given surface S one selects a one-parameter family of lines and subjects the tangent strip along each line to a transformation depending on the same parameter. If the transformed strips generate a new surface S^*, one says that S^* is generated by the lines of S. The

present paper considers the nature of the lines that may be chosen on S if the transformation is, first, projective, in which case the lines must be plane sections of S, second, equiaffine, in which case the class of lines is subject to only one condition. Families of planes or cylindrical lines belong to this class. Finally, if the transformation is a parallel displacement, the generating lines must be cylindrical.

M. S. Knebelman (Pullman, Wash.).

Berezina, L. Ya. Some properties of evolute surfaces. Uspehi Matem. Nauk (N.S.) **8**, no. 3(55), 109–110 (1953). (Russian)

Let R_1, R_2 be the radii of principal curvature of a surface (Σ), R_1^i, R_2^i those of its evolute surfaces (Σ_i), $i=1, 2$, and β_i the angle between the normal to (Σ) and the lines of curvature of (Σ_i); then

$$\sin 2\beta_1 \sin 2\beta_2 = \frac{4(R_1-R_2)^2}{(R_1^1-R_2^2)(R_1^2-R_2^1)},$$

$$(R_2^1-R_1^1)\sin 2\beta_1 = 2(R_1-R_2)\frac{\tan\psi_1}{\tan\psi_2},$$

$$(R_2^2-R_1^2)\sin 2\beta_2 = 2(R_1-R_2)\frac{\tan\psi_2}{\tan\psi_1}.$$

Here $\tan\psi_i = R_i/\rho_{ig}$ (ρ_{1g}, ρ_{2g} = radii of geodesic curvature of the lines of curvature on Σ). The β_i satisfy the equation

$$R_1^i \sin^2\beta_i + R_2^i \cos^2\beta_i = k_n^i/K_i,$$

where k_n^i is the curvature of the normal section to (Σ_i) in the direction of the normal to Σ; K_i is the Gaussian curvature of (Σ_i). *D. J. Struik* (Cambridge, Mass.).

Bol, G. Über die Flächen, deren Godeaux-Kette sich schliesst und die Periode 8 hat. Arch. Math. **4**, 61–74 (1953).

It is known that associated with a surface in a projective space of three dimensions there is a Godeaux sequence of conjugate nets. In this paper, the author uses the half-invariant differentiation to study the surfaces whose Godeaux sequence is periodic with a period eight, and calls these surfaces octagon surfaces. An octagon surface is called an octagon minimal surface if it is also a projective minimal surface. All octagon surfaces are determined by six arbitrary functions of one variable, but all octagon minimal surfaces by two arbitrary functions of one variable. Among the octagon minimal surfaces the simplest one is Carton's surface. Moreover, the diagonals of all the quadrilaterals of Demoulin associated with any octagon minimal surface belong to the same linear congruence, but the octagon minimal surfaces are not the only surfaces having this property. *C. C. Hsiung* (Bethlehem, Pa.).

***Strubecker, Karl. Le superficie, le cui asintotiche dei due sistemi appartengono a complessi lineari.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 441–445. Casa Editrice Perrella, Roma, 1953.

Let Φ be a surface in a three-dimensional space such that each family of its asymptotic tangents belongs to a linear complex. It is known that all such surfaces Φ can be divided into three distinct classes I, II and III. In this paper, simple analytic representations are obtained for any surfaces Φ_{I}, Φ_{II}, Φ_{III} belonging to these classes I, II, III and in elliptic, quasi-elliptic and isotropic spaces, respectively. This result is contained in a previous paper by the author [Math. Z. **52**, 401–435 (1949); these Rev. **11**, 459]. *C. C. Hsiung.*

Kruppa, Erwin. Eine projektive Invariante von drei Linienelementen n-ter Ordnung mit einem gemeinsamen Linienelement $(n-1)$-ter Ordnung. Monatsh. Math. **57**, 134–139 (1953).

Let C_i $(i=1, 2, 3)$ be three curves in the (x, y)-plane having a contact of order $n-1$, but not n, at the origin P $(0, 0)$ with the equations

$$y = C_2x^2 + C_3x^3 + \cdots + C_{n-1}x^{n-1} + C_n x^n + \cdots.$$

In this paper, the author derives the projective invariants

$$J^{(n)} = (C_{n,3}-C_{n,1})/(C_{n,3}-C_{n,2}), \quad (n>1),$$

and characterizes them affinely for $n=3$, 4 and any $n>4$ and projectively for $n=6$ and 7. It is also shown that $J^{(3)} = (\kappa_3-\kappa_1)/(\kappa_3-\kappa_2)$, where κ_i is the curvature of C_i at P.

C. C. Hsiung (Bethlehem, Pa.).

Bereis, R. Eine Bemerkung zur projektiven Invariante J^n in der voranstehenden Arbeit von E. Kruppa. Monatsh. Math. **57**, 140–141 (1953).

The general projective invariant $J^{(n)}$ defined in the paper reviewed above is geometrically interpreted as follows:

$$J^{(n)} = [M_{2,n-1}-M_{1,n-1}]/[M_{3,n-1}-M_{2,n-1}],$$

where $M_{i,n-1}$ is the radius of curvature at the point corresponding to P of the $(n-2)$th evolute of C_i for $n>2$. A new geometric interpretation of $J^{(4)}$ is also given.

C. C. Hsiung (Bethlehem, Pa.).

***Hlavatý, Václav. Differential line geometry.** Translation by Harry Levy. P. Noordhoff Ltd., Groningen, 1953. x+495 pp. 22.50 florins; cloth, 25.00 florins.

A translation of the author's Differentielle Liniengeometrie [Noordhoff, Groningen, 1945; these Rev. **8**, 346]. Minor corrections and modifications have been made in preparing the translation.

***Šulikovskiĭ, V. I. The theory of nets and some questions of classical differential geometry.** Sto dvadcat' pyat' let neevklidovoĭ geometrii Lobačevskogo, 1826–1951 [One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ, 1826–1951], pp. 201–205. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 7.60 rubles.

This is a report on certain results concerning nets on two-dimensional manifolds with an affine or Riemannian connection, generalizing work of N. Efimov [Trudy Sem. Vektor. Tenzor. Analizu **5**, 148–172 (1941); these Rev. **8**, 346]. Considered are a triplet of nets $\overset{\alpha}{A}$, $\alpha=1, 2, 3$, mutually apolar, and their connections $\overset{\alpha}{G}$ apart from the initial connection; these connections are conjugate with respect to the triplet $\overset{\alpha}{A}$ in the sense of A. P. Norden [Spaces of affine connection, Gostehizdat, Moscow-Leningrad, 1950; these Rev. **12**, 441]. Here the Čebyšev vector, introduced by Ya. S. Dubnov, plays an important role. Among the cases considered are nets apolar to $\overset{1}{A}$, surfaces in projective P_3 admitting ∞^1 conjugate sets with general lines of Green, and surfaces of Voss. *D. J. Struik* (Cambridge, Mass.).

Hartman, Philip, and Wintner, Aurel. On asymptotic parametrizations. Amer. J. Math. **75**, 488–496 (1953).

In an earlier paper [same J. **73**, 149–172 (1951); these Rev. **12**, 742], the authors have shown that if a surface S of class C^n $(n\geq 2)$ possesses an asymptotic C^n-parametrization, then it possesses C^{n+1}-parametrizations. In the present

paper, the converse question is given the following answer: If S possesses a C^m-parametrization where $m \geqq 2$, and if the Gaussian curvature $K(<0)$ is a function of class C^{m-1} in this parametrization, then S possesses asymptotic C^{m-1}-parametrizations. It is also shown that if $m=2$ the restriction placed on the class of K (or an equivalent condition) is a necessary condition for the conclusion to obtain.

A. Fialkow (Brooklyn, N. Y.).

Makarova, Z. T. Investigation of an integer-valued invariant of N binary forms of degree $n>4$. Mat. Sbornik N.S. **33(75)**, 233–240 (1953). (Russian)

The paper arrives by a different method at substantially the same result as Tartakovskiĭ in Mat. Sbornik (N.S.) **32(74)**, 225–248 (1953); these Rev. **14**, 687.

H. Busemann (Los Angeles, Calif.).

Aruffo, Giulio. Sul differenziale generalizzato delle forme differenziali esterne. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Fis. Mat. Nat. (8) **13**, 367–372 (1952); **14**, 13–18 (1953).

If ω_r denotes an exterior differential form of degree r defined in an n-cell C_n, the theorem of Stokes tells us that

$$\int_{\partial D_{r+1}} \omega_r = \int_{D_{r+1}} \omega_{r+1} \tag{1}$$

for all smooth $(r+1)$-chains D_{r+1} in C_n, where ω_{r+1} is the exterior derivative $d\omega_r$ of ω_r, provided that ω_r is of class r $(r \geqq 1)$. B. Segre [Forme differenziali e loro integrali, v. I, Edizioni Univ. Docet, Roma, 1951, pp. 186–200; these Rev. **14**, 208] has given certain necessary and sufficient criteria for the existence of a continuous form ω_{r+1} satisfying (1), when ω_r is merely continuous. This paper gives a simplified version of Segre's criteria, the most important result being as follows: if $\omega_r^{(m)}$ is the approximation to ω_r obtained by replacing the coefficients of ω_r by their mth polynomial approximation of Stieltjes in C_n, divided by the mth polynomial approximation of unity, then a necessary and sufficient condition for (1) to hold is that $d\omega_r^{(m)} \underline{m} \omega_{r+1}$ in C_n. Some applications are made; in particular, it is shown that if the coefficient of $dx^{i_1} \cdots dx^{i_r}$ in ω_r has continuous derivatives with respect to x_j, for all $j \neq i_1, \cdots, i_r$, and for all choices of $i_1, \cdots, i_r$, then ω_{r+1} can be computed as in the case in which ω_r is of class 1.

W. V. D. Hodge.

Ishii, Yoshihito. On the Riemannian space with a discontinuous metric. Tensor (N.S.) **3**, 13–22 (1953).

This paper studies the geometry of a space which is the union of two Riemann spaces having a common boundary. This space is termed a Riemann space with a discontinuous metric. Parallelism of vectors, tensor differentiation and the curvature tensor are defined on the common boundary by generalization of the corresponding concepts for a Riemann space. For the two-dimensional case, a generalized Gauss-Bonnet formula is derived. The properties of the curvature tensor are studied in some detail, particularly when the space is four-dimensional.

A. Fialkow.

Yen, Chih-Ta. Sur la connexion projective normale associée à un feuilletage du deuxième ordre. Ann. Mat. Pura Appl. (4) **34**, 55–94 (1953).

In an n-dimensional number space R^n a k-dimensional lamination (feuilletage) of the 2nd order is defined as a system of k-dimensional sub-spaces (laminae = feuilles) such that, for every point P of R^n and for every k-plane at P, there is a member of the system which passes through P and is tangent to the k-plane. It is proved that this structure admits a unique semi-normal projective connexion (strictly, a connexion in the manifold $R^{n,k}$ of k-planes at points of R^n) with respect to which the laminae are k-geodesics. This connexion satisfies certain other invariant conditions and is called the normal projective connexion associated with the 2nd order lamination. For $k=n-1$ it is the connexion defined by Hachtroudi [Les espaces d'éléments à connexion projective normale, Hermann, Paris, 1937].

It is shown that a given connexion is not necessarily the normal projective connexion associated with some lamination; for this it must satisfy certain conditions which arise as the conditions for complete integrability of the equations for k-geodesics. For $k>1$ these conditions are given explicitly. It is also proved that a k-dimensional 2nd order lamination $(k>1)$ is equivalent to a linear lamination in projective space if and only if it admits a group of automorphisms whose isotropy group is k-transitive (i.e., leaves a point 0 invariant and carries any k-plane at 0 into any other k-plane at 0).

There is also defined a unimodular affine connexion of k-planes (affine connexion in $R^{n,k}$), and it is proved that there is such a connexion associated with any given 2nd order lamination; it has zero torsion, and the laminae become k-geodesics. A number of special cases are examined.

A. G. Walker (Liverpool).

Liber, A. E. On the theory of surfaces in projective space. Doklady Akad. Nauk SSSR (N.S.) **90**, 137–140 (1953). (Russian)

The manifold of all one-dimensional subspaces B_1 of an $(n+1)$-dimensional vector space B_{n+1} is called an $(n+1)$-dimensional pseudo-vector-space $\mathfrak{B}_{n+1}$. Every B_1 is uniquely determined by an element $\mathfrak{X}$. A basis in B_{n+1} is determined by $\mathfrak{X}^i$, $i=1, 2, \cdots, n+1$, which transform as the components of a contravariant vector in B_{n+1} and in B_1. A space B_{n+1} is isomorphic to a central-affine E_{n+1} and a subspace B_1 to an E_1, which is a line passing through the center of E_{n+1}. The pseudo-vector space $\mathfrak{B}_{n+1}$ can therefore be mapped on the manifold of all central lines of E_{n+1}. We can consider the pseudo-vector $\mathfrak{X}^i$ as the connecting affinor of the two central-affine spaces E_{n+1} and E_1 in the sense of V. V. Vagner in his introduction to the translation of Veblen-Whitehead's "Foundations of differential geometry" [Izdat. Inostn. Lit., Moscow, 1949]. With the aid of these concepts the differential geometry of m-dimensional manifolds S_m in a projective space P_n can be developed; the arithmetization of such a P_n is realized by a one-to-one mapping on $\mathfrak{B}_{n+1}$. To every point η^a $(a=1, \cdots, m)$ of the S_m belongs a radial E_1 isomorphic to B_1 determined by $\mathfrak{X}(\eta^a)$. The tangent B_m is defined by m independent pseudo-vectors $\mathfrak{X}_a = \partial_a \mathfrak{X} + \gamma_a \mathfrak{X}$; when $B_{n+1} = B_1 + B_m + B_{n-m}$, then B_{n-m} gives the rigging. The resulting equations fall into two groups:

$$D_a \mathfrak{X} = \mathfrak{X}_a, \quad D_a \mathfrak{X}_b = h^p_{ab} \bar{n}_p + g_{ab} \mathfrak{X}, \quad D_a \bar{n}_p = h^c_{ap} \mathfrak{X}_c + w_{ap} \mathfrak{X},$$

and

$$\bar{R}_{ab} = 2g_{[ab]}, \quad h^p_{[ab]} = 0, \quad D_{[a} h^p_{b]c} = 0,$$

$$R^d_{abc} = 2h^p_{[a|c|} h^d_{b]p} - 2\delta^d_{[a} g_{b]c} + 2g_{[ab]} \delta^d_c, \quad D_{[a} h^c_{b]p} + \delta^c_{[a} w_{b]p} = 0,$$

$$R^e_{abp} = 2h^e_{[a|p|} h^p_{b]e}, \quad D_{[a} g_{b]c} + h^p_{[b|c|} w_{a]p} = 0, \quad D_{[a} w_{b]p} + h^e_{[b|p|} g_{a]e} = 0,$$

where the $\bar{n}_p$ form a basis for the rigging and $\bar{R}_{ab}$, $R^{\cdots d}_{abc}$ and $R^{\cdots e}_{abp}$ are the curvature affinors in the radial, tangent and rigging composite manifolds. If the γ_a, Γ^c_{ab}, Γ^e_{ap} and the h^p_{ab}, h^c_{ap}, g_{ab}, w_{ap} are given and satisfy the given conditions,

then the S_m in P_n is determined but for automorphisms of P_n. It is further shown how an invariant rigging can be obtained.

D. J. Struik (Cambridge, Mass.).

Laptev, G. F. Differential geometry of imbedded manifolds. Group theoretical method of differential geometric investigations. Trudy Moskov. Mat. Obšč. **2**, 275–382 (1953). (Russian)

As the subtitle of this paper indicates, the method of investigation of subspaces is group-theoretic. For this reason the first part of the paper is a resumé of the essential properties of Lie groups. In particular, the fundamental differential forms of Cartan θ^s and $\bar{\theta}^s$, defined by

$$-\theta \circ v = v + dv = v \circ \bar{\theta}^s,$$

are used to obtain the equations of Lie. This group G, of r parameters, is realized by means of an n-dimensional analytic coordinate space. Let H be a fixed subgroup of G having no invariant subgroup in common with G (or if G and H have a maximal invariant subgroup D, one takes the factor groups G/D, H/D). This subgroup H is the "group of support" and has an image F in the realization space, H being stationary for F. A geometrical object is then defined as a point in any realization space $\mathcal{S}$ of the group of support, whose coordinates are referred to the coordinate frames of F. This definition agrees with that of Veblen and Whitehead, but it is constructive in the sense that the components (coordinates of a point in $\mathcal{S}$) are given as solutions of systems of total differential equations. It is in this manner that generalizations of tensors, affinors, pseudo-tensors are constructed. The last two chapters are devoted to the study of the geometrical objects of a subspace, but beyond the general relations and conditions that they must satisfy, the results are meager. Only in the case of the projective differential geometry of a surface is the work complete and all the classical invariants are obtained from this general principle. In conclusion, the author states that the methods developed can be applied equally well to affine and conformal geometry of subspaces, but that the algebraic difficulties become unresolvable. In the reviewer's opinion, the main value of this group-theoretic method is not so much as a means of investigating any particular geometry, but as a unifying principle for all of modern differential geometry.

M. S. Knebelman (Pullman, Wash.).

Wong, Yung-Chow. A class of non-Riemannian K^*-spaces. Proc. London Math. Soc. (3) **3**, 118–128 (1953).

Recurrent spaces of symmetric affine connection are defined by equations analogous to those for recurrent Riemannian spaces, and a special sub-class of these spaces is defined in terms of a privileged coordinate system. A complete classification of the special recurrent spaces is given, with appropriate canonical forms, and it is found which of these spaces admit a Riemannian structure. Comparisons are made with results given by Ruse and the reviewer, and it is shown that the general canonical form given by the reviewer [same Proc. (2) **52**, 36–64 (1950); these Rev. **12**, 283] can be simplified to involve fewer arbitrary functions. The number of independent parallel vector fields for each type of spatial recurrent space is calculated.

A. G. Walker (Liverpool).

Tandai, Kwoichi. On areal spaces. VI. On the characterization of metric areal spaces. Tensor (N.S.) **3**, 40–45 (1953).

In this paper the author proves a theorem which has been considered in the series of papers on areal spaces of which this is the latest. The problem concerns the deduction of a two-index metric tensor where the fundamental invariant of the geometry is the multiple integral $\int F(x, \partial x/\partial u)du^1 \cdots du^n$ in n-dimensional space. If the function F satisfies the conditions which make the integral independent of the parametric representation, it can be the fundamental function of a Finsler space for $m=1$, and of a Cartan space based on the notion of area for $m=n-1$. In both these cases the deduction of the two-index metric tensor is immediate. For $1<m<n-1$ the problem is more difficult. For $m=2$, for instance, the function F can be expressed as a function $L(x, p^{ij})$ of x and a bivector p^{ij}. This can lead easily to a 4-index metric tensor $g_{i_1i_2,j_1j_2}$ which is suitable for the measurement of bivectors. If this 4-index tensor can be split up into two-index tensors with the relation $g_{i_1i_2,j_1j_2}=2g_{[i_1[j_1}g_{j_2]i_2]}$ satisfied, the original space is said to belong to the metric class. The main result of this paper is that Riemannian, Finsler, and Cartan spaces are the only spaces of the metric class.

E. T. Davies (Southampton).

Tonooka, Keinosuke. Generalized rheonomic geometry of K-spreads. Tensor (N.S.) **3**, 26–39 (1953).

This paper is concerned with a two-fold generalization of the geometry of K-spreads as studied by Douglas [Math. Ann. **105**, 707–733 (1931)]. In the first place the second-order system of differential equations is replaced by equations of order $m>3$. In the second place the group of transformations considered is the group of generalized rheonomic transformations

$$(1) \qquad x^{i'}=x^{i'}(x, u), \quad u^{\alpha'}=u^{\alpha'}(u^\alpha)$$

(Latin indices run from 1 to n; Greek indices from 1 to K).

If $x^i=x^i(u^\alpha)$ are the parametric equations of a K-dimensional surface, the set of functions

$$(2) \qquad x^i(\alpha), \quad p^i_{\alpha(1)}=\frac{\partial x^i}{\partial u^{\alpha_1}}, \cdots, p^i_{\alpha(m)}=\frac{\partial^m x^i}{\partial u^{\alpha_1}\cdots\partial u^{\alpha_m}}$$

characterize the surface element of the mth order of the surface. In this notation the geometry of K-spreads of higher order can be written

$$(3) \qquad p^i_{\alpha(m)}+H^i_{\alpha(m)}(u^\beta, x^j, p^j_{\beta(1)}, \cdots, p^j_{\beta(m)})=0, \quad m>3.$$

The author starts by quoting the laws of transformation of the functions (2), and the consequent law of transformation of the H functions occurring in (3).

The remainder of the paper is chiefly concerned with the determination of three-index coefficients of affine connection from the H functions. These coefficients can have indices which are all Latin or all Greek, or partly Latin and partly Greek. The author considers the $(n+K)$-dimensional manifold X_{n+K} with point coordinates $x^1, \cdots, x^n, u^{\dot{1}}, u^{\dot{2}}, \cdots, u^{\dot{K}}$. On putting for convenience $y^A=x^i$ $(A=1, \cdots, n)$, $y^A=u^\alpha$ $(A=\dot{1}, \dot{2}, \cdots, \dot{K})$, the generalized rheonomic group (1) may be regarded as the group of coordinate transformations $y^{A'}=y^{A'}(y^A)$ with $\partial y^{A'}/\partial y^A=\partial x^{i'}/\partial x^i$ $(A'=i', A=i)$, $=\partial x^{i'}/\partial u^\alpha$ $(A'=i', A=\alpha)$, $=0$ $(A'=\alpha', A=i)$, $=\partial u^{\alpha'}/\partial u^\alpha$ $(A'=\alpha', A=\alpha)$. He then writes down the classical law of transformation of connection coefficients Γ^A_{BC} giving them the special forms indicated by the special values of these derivatives $\partial y^{A'}/\partial y^A$ just given. Covariant derivatives of the different possible types are given. The paper ends with a study of the equivalence problem under this group of transformations.

E. T. Davies (Southampton).

Kawaguchi, Michiaki, Jr. On some theorems in the *G*-extensor analysis. Tensor (N.S.) **3**, 46–52 (1953).

The author introduced the *g*-extensor or generalized extensor in a previous paper [Tensor (N.S.) **2**, 59–66 (1952); these Rev. **14**, 690]. In the present paper, various *g*-extensors are determined from ordinary extensors by means of differentiation and contraction and use of new ranges. For example, many new covariant and contravariant *g*-extensors of reduced range are constructed. Some of these are obtained from the expoint and functions of the expoint. Finally, generalizations of the *g*-extensor and related invariants are obtained by repeated use of the $\int$, y, and z operations of extensor analysis [A. Kawaguchi, J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9**, 1–152 (1940); these Rev. **2**, 22]. *N. Coburn* (Ann Arbor, Mich.).

Sasayama, Hiroyoshi. On extensors in the space of non-integral order. Tensor (N.S.) **3**, 53–64 (1953).

In this paper, the extensor concept is extended to spaces of non-integral order by use of fractional differentiation. First, by use of a Leibnitz type of rule for fractional differentiation of a product, the various properties of the extended coordinate transformation of fractional order are studied. The infinite series encountered are assumed to converge. Further, some properties of the fractional derivatives of weighed Jacobians of the coordinate transformation are determined. Then the author introduces excovariant and excontravariant vectors of fractional grade by use of transformation formulas involving coordinate transformations of fractional order, and also Jacobian exvectors by use of a similar scheme. The remainder of the paper is concerned with determining exvectors and invariants associated with a given exvector.

N. Coburn (Ann Arbor, Mich.).

Suguri, Tsuneo. Errata: Theory of invariants in the geometry of paths. J. Math. Soc. Japan **5**, 112a (1953). See same J. **4**, 231–268 (1952); these Rev. **14**, 902.

NUMERICAL AND GRAPHICAL METHODS

***Table of natural logarithms for arguments between zero and five to sixteen decimal places.** National Bureau of Standards Applied Mathematics Series, No. 31. U. S. Government Printing Office, Washington, D. C., 1953. x+501 pp. $3.25.

Reprint of a volume first issued in 1941 [cf. these Rev. **3**, 152].

***Kartografičeskie tablicy.** [Cartographic tables.] 2d ed. Trudy Central'nogo Naučno-Issledovatel'skogo Instituta Geodezii, Aèros"emki i Kartografii, vyp. 97. Izdat. Geodez. i Kartograf. Lit., Moscow, 1953. 137 pp. 14.00 rubles.

Ishiguro, Eiichi. Tables useful for the calculation of the molecular integrals. IV. Nat. Sci. Rep. Ochanomizu Univ. **4**, 64–76 (1953).

[For parts I–III see these Rev. **14**, 799.] This paper contains 12 pages of numerical tables of the integral

$$W_r^{\rho}(m, n; \alpha, \beta) = \int_1^{\infty}\int_1^{\infty} Q_r^{\rho}(\lambda_+)P_r^{\rho}(\lambda_-)e^{-\alpha\lambda_1-\beta\lambda_2} \times \lambda_1^m\lambda_2^n(\lambda_1^2-1)^{\rho/2}(\lambda_2^2-1)^{\rho/2}d\lambda_1 d\lambda_2$$

where λ_+ and λ_- are, respectively, max and min of λ_1, λ_2. The values of the parameters have been chosen so as to facilitate calculation of exchange and hybrid integrals for the normal Li_2 molecule, and the author hopes that "these tables will serve as the pilots for the more complete tables of the two-parameter W_r^{ρ} functions". *A. Erdélyi.*

Russek, Joy, and Twersky, Vic. Graphs of the function

$$E(N, \delta) = \sum_{n=1}^{N} n^{-1/2}e^{in\delta}.$$

Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-49**, i+22 pp. (1953).

The authors present 34 graphs pertaining to the function mentioned in the title. Let $E=C+iS$. Curves S versus C are given for $N=1(1)10$, $0\leq\delta\leq180°$; C and S as functions of δ for the same range of variables. There are vector plots of E for δ constant and N increasing from 1 to 10 ($\delta=10, 30, 90, 110°$). The remaining graphs show C versus S as functions of $u=(2N\delta/\pi)^{\frac{1}{2}}$ for $N=10^2$, 10^3, and 10^4, and the functions C and S for the same range of u and N. The values of the graphs are accurate to about five percent. Details of computation are given. The function E occurs in multiple-scattering and antenna problems. *C. J. Bouwkamp.*

Uhler, Horace S. Exact location of the 10^mth digit in the consecutively written sequence of the natural numbers 1, 2, 3, 4, ⋯. Scripta Math. **19**, 201–204 (1953).

Siegel, K. M., Crispin, J. W., Kleinman, R. E., and Hunter, H. E. Note on the zeros of $(dP^1_{n_i}(x)/dx)|_{x=x_i}$. J. Math. Physics **32**, 193–196 (1953).

Continuing an earlier paper [same J. **31**, 170–179 (1952); these Rev. **14**, 321] the authors describe the computation, for given x, of positive values n_i for which $dP_n^1(x)/dx=0$, and the computation of the corresponding integrals $\int_x^1[P_{n_i}(t)]^2dt$. They illustrate their method by computing approximations for these quantities when $x=\cos 165°$ and $i=1(1)15$. *A. Erdélyi* (Pasadena, Calif.).

Ceschino, F. Sur la résolution des équations par la méthode de Lin. Ann. Soc. Sci. Bruxelles. Sér. I. **67**, 77–82 (1953).

Lin [J. Math. Physics **20**, 231–242 (1941); these Rev. **3**, 153] suggested a convenient iterative method for finding the real roots of a polynomial equation. This paper shows that the method has the following geometric interpretation: Let β_1 be an approximation to a root of $f(x)=0$, then the next approximation is the abscissa of the intersection with the x-axis of the line through the points 0, $f(0)$ and $\beta_1 f(\beta_1)$, etc. *B. Friedman* (New York, N. Y.).

Aldrich, Leland E. Solution of algebraic equations. J. Franklin Inst. **256**, 59–69 (1953).

A method is described for the solution of polynomial equations with real coefficients. *E. Frank.*

Grinstein, Louisa S. Upper limits to the real roots of polynomial equations. Amer. Math. Monthly **60**, 608–615 (1953).

Rules are summarized for the computation of upper bounds to the real roots of polynomial equations with real coefficients. *E. Frank* (Chicago, Ill.).

Sarafyan, Diran. Nested series, computation of square roots and solution of third degree equations. Math. Mag. **27**, 19–36 (1953).

If $y=cf(x)$, where c is a real constant number, a nested series with infinitely many terms is defined by

$$cf\{x+cf[x+cf(x)]\}\cdots.$$

Such a nested series sometimes converges to a limit on some interval. The meeting point of two objects or points in motion is represented by the limit of a nested series. A geometric interpretation is given. It is shown that the equations $x^n+Ax^{n-1}+B=0$ and $x^n+Ax+B=0$ have real roots given by the limit of nested series. Third degree equations in particular are discussed. *E. Frank* (Chicago, Ill.).

Baur, Arnold. Die näherungsweise Lösung von Gleichungen. Math.-Phys. Semesterber. **3**, 235–237 (1953).

Antosiewicz, H. A., and Hammersley, J. M. The convergence of numerical iteration. Amer. Math. Monthly **60**, 604–607 (1953).

Hoffman, A., Mannos, M., Sokolowsky, D., and Wiegmann, N. Computational experience in solving linear programs. J. Soc. Indust. Appl. Math. **1**, 17–33 (1953).

Three methods of solving certain problems in linear programming were carried out on the SEAC. These are Dantzig's "Simplex" method, Brown's Fictitious Play method, and Motzkin's Relaxation method. The simplex method is more complicated algebraically than the other two. Nevertheless even when this one required the use of the external memory and the others did not, it required considerably less machine time for the same degree of accuracy.

A. S. Householder (Oak Ridge, Tenn.).

***Stiefel, Eduard. Some special methods of relaxation technique.** Simultaneous linear equations and the determination of eigenvalues, pp. 43–48. National Bureau of Standards Applied Mathematics Series, No. 29, U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

The author shows that the well-known relaxation techniques for solving simultaneous linear equations (with positive definite symmetric matrix) can be considered as methods of descent. He then exhibits a particular method of descent for which "the solution of the n equations in n unknowns will be found in precisely n steps". This method is the forerunner of the conjugate gradient method, which has been inspired recently by the independent work of Hestenes, Lanczos and Stiefel. The author describes two examples in which the finite difference equations for $\Delta u=0$ and $\Delta\Delta u=0$ are solved by his n-step method of relaxation. The latter problem arose in the study of the stresses present in a dam and required the solution of 139 simultaneous equations.

E. Isaacson (New York, N. Y.).

Wang, Chi-Teh, and DeSanto, Daniel F. Differentiation of experimental data by means of higher order finite-difference formulas. J. Aeronaut. Sci. **20**, 792–793 (1953).

Phillips, William. Approximate integration. J. Inst. Actuar. **77**, 159–179; discussion, 180–195 (1951).

Eltermann, Heinz. Ein Beitrag zur numerischen Integration bei nicht gleichabständigen Abszissen und zur Berechnung von Kurvenintegralen. Z. Angew. Math. Mech. **33**, 254–255 (1953).

For numerical integration in the case of unequally spaced abscissas the author proposes to make the independent variable x a function of a new parameter t, where the function is to be chosen so that the values of t giving the required values of x are equally spaced. Some illustrative examples are given for the case where polynomial functions are used to relate $f(x)$ to x and other polynomial functions used to relate x to t. No analysis of error is furnished.

W. E. Milne (Corvallis, Oregon).

Reich, E. A random walk related to the capacitance of the circular plate condenser. Quart. Appl. Math. **11**, 341–345 (1953).

Consider the discrete one-dimensional random walk for which the probability is $dt/[1+(x-t)^2]$ that a particle at x moves, in one step, into the interval $t\leqq x\leqq t+dt$. Let the particle start from a point x in $-1/\kappa\leqq x\leqq 1/\kappa$ and denote by $f(x)$ the expected number of steps until the particle leaves this interval for the first time. With the help of a theorem of E. R. Love [Quart. J. Mech. Appl. Math. **2**, 428–451 (1949); these Rev. **11**, 629] it is shown that the capacitance C of a condenser consisting of two circular discs of radius a and distance κa satisfies the relation $C=(a\kappa/2\pi)\int_{-1/\kappa}^{1/\kappa}f(t)dt$. This result leads to a sampling method for the numerical calculation of C whose practicality is discussed. The probabilistic interpretation yields also some information on the asymptotic behavior of C, as $\kappa\to 0$, or $\kappa\to\infty$.

W. Wasow (Los Angeles, Calif.).

Rapoport, I. M. A new method of approximate integration of ordinary differential equations. Ukrain. Mat. Žurnal **4**, 399–413 (1952). (Russian)

The author presents a method for the approximate integration of ordinary differential equations applicable to differential equations of arbitrary order, and not requiring any special limitations. It is shown that the accuracy of the mth approximation, obtained by this method, in the instance of a sufficiently good initial approximation, will not exceed $K\theta^{m^m}$ $(\theta<1)$ where K and θ are constants independent of m and can be calculated from the initial approximation. Part 1 treats linear differential equations, in which case a sharper error estimate is established. In part 2 an iteration process is set up for nonlinear differential equations.

W. E. Milne (Corvallis, Ore.).

Sterne, Theodore E. The accuracy of numerical solutions of ordinary differential equations. Math. Tables and Other Aids to Computation **7**, 159–164 (1953).

In the numerical solution of a system of ordinary differential equations, linear or non-linear, errors grow in accordance with the solution of the variational equations $\dot{x}_i=\sum_{j=1}^{n}a_{ij}x_j+b_i$, $i=1, 2, \cdots, n$, which are effectively the linear equations governing the correction to a first approximation to the required solution. If the b_i are introduced through rounding errors in the x_i, the error in x_k at point t_1, caused by errors ϵ_i at point t_0, is $\sum_{i=1}^{n}\epsilon_i\lambda_i$, where λ is that solution of the adjoint equations $\dot{\lambda}_i=-\sum_{j=1}^{n}a_{ji}\lambda_j$ which has the initial conditions $\lambda_k(t_1)=1$, $\lambda_j(t_1)=0$, $j\neq k$. If errors are introduced at each point in the range t_0 to t_1, the total error

in x_k becomes $\sum_0^{t_1}\sum_1^n \epsilon_i(t)\lambda_i(t)$, and formulae are given for the variance of these results according as the errors are random or biassed. If the b_i are introduced through truncation error, their magnitudes can be estimated from the first approximate solution, and $x_k(t)=\int_0^t\sum_{i=1}^n b_i(t)\lambda_i(t)dt$. The use of the method in deciding the type of integration formulae, size of interval and number of working figures needed to obtain a result of specified precision is illustrated by application to a non-linear first-order equation with a known analytical solution. This paper appears to be the first published account of a type of method of error assessment of great value and easy application. *L. Fox.*

Laudet, Michel. Intégrations numériques de l'équation des trajectoires électroniques. J. Phys. Radium (8) **14**, 604–610 (1953).

Southwell, R. V. Some extensions of 'Rayleigh's principle'. Quart. J. Mech. Appl. Math. **6**, 257–272 (1953).

An extension is given of "Rayleigh's Principle" applicable to systems having eigenvalues which are both positive and negative. In conjunction with intensification by iteration, the extension permits determination of upper and lower bounds to the eigenvalue of the gravest mode. A theorem, due to G. Vaisey, is given which shows that the approximate eigenvalue for one iteration cannot lie between the least positive and least negative eigenvalues. An example concerning the elastic stability of a flat-plate cantilever under load is presented. In this problem, eigenvalues having both signs are present; the upper and lower limits are within about 1 percent of each other. *S. Levy.*

Lyusternik, L. A. Application of cubature formulas to numerical solution of Cauchy's problem for certain partial differential equations. Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 178–181 (1953). (Russian)

Let Q_c be the normed linear space of all continuous functions $\varphi(A)$ defined at all points A of the (x, y)-plane which grow more slowly than $\exp c|A|^2$ for $c>0$ as $|A|^2\to\infty$, and let $\|\varphi\|_c=\max|\varphi(A)\exp(-c|A|^2)|$. Consider the operator $u(t, l)$ defined by

$$u(t, l)\varphi(A)=c_0\varphi(A)+\sum_1^l c_i S[2t^{\rho_i}, 4l+2]\varphi(A).$$

Here $S[\rho, n]\varphi(A)$ is the arithmetic mean of the values of φ at the vertices of a regular polygon of n sides inscribed in a circle of radius ρ, center A. The positive numbers c_i and ρ_i satisfy $c_0+\sum_1^l c_i=0$ and $\sum_1^l c_i\rho_i^{2s}=s!$, $1\leqq s\leqq 2l$. The author shows that if $\varphi\in Q_c$ and $0<t<1/8c$, then as $l\to\infty$, $u(t, l)\varphi(A)$ converges, in the sense of the metric of Q_{2c}, to the solution of the heat equation $\partial v/\partial t=\Delta v$ with $v(x, y, 0)=v(A, 0)=\varphi(A)$. At all finite points A at any time t in the interval $[t_0, t_1]$ with $0<t_0<t_1<1/8c$ the convergence is uniform.

J. H. Giese (Havre de Grace, Md.).

Macneal, R. H. An asymmetrical finite difference network. Quart. Appl. Math. **11**, 295–310 (1953).

An electrical interpretation of partial differential equations of the form $\nabla\cdot(\sigma\nabla\phi)+\tau=0$ in two variables is used as a guide to determine the parameters of a network whose graph is a triangulation of the region in which the equation is to hold. The nodal equations of this network are the difference equations used to replace the partial differential equations. The flexibility of this method simplifies the problems involved in dealing with curved boundaries and with variable mesh sizes for a given system. Also, the difference equations of this method can be directly realized by an electrical network analog computer. The method is applied to the computation of the TEM modes of a conical line resonator, and dual networks of the derived networks are discussed. *C. Saltzer* (Cleveland, Ohio).

***Synge, J. L. Flow of viscous liquid through pipes and channels.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 141–165. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

The author presents a numerical procedure for determining the function $u(x, y)$ which satisfies the Laplace equation $u_{xx}+u_{yy}=0$, for (x, y) in the region D, and such that u takes on prescribed values on part of the boundary B_1 while the normal derivative of u takes on given values on the rest of boundary B_2. Such problems are exemplified by (1) a discussion of incompressible, steady, laminar, viscous flow through a pipe, and (2) a treatment of water currents produced in a channel by wind drag.

The author obtains an approximate solution by applying his technique of the hypercircle with the use of special "pyramid functions". This requires the iterative solution of what look like linear finite difference equations. He employs "quick-convergence" factors to speed the iterative process. The author obtains a bound for the mean square deviation of the gradient of his approximate solution, as well as bounds for the Dirichlet integral of the true solution.

E. Isaacson (New York, N. Y.).

Mignot, Noël. Sur les solutions numériques du problème de la chaleur. I. C. R. Acad. Sci. Paris **236**, 1735–1737 (1953).

The author obtains preliminary results with regard to the definiteness and selfadjointness of a class of finite-difference formulations simulating a generalized one-dimensional heat-flow problem and states that the significance of these results is to be established in a later paper.

F. B. Hildebrand (Cambridge, Mass.).

Mignot, Noël. Sur les solutions numériques du problème de la chaleur. II. C. R. Acad. Sci. Paris **236**, 2375–2377 (1953).

Stability criteria [in the sense defined by O'Brien, Hyman and Kaplan, J. Math. Physics **29**, 223–251 (1950); these Rev. **12**, 751] are derived for various finite-difference representations of the heat-conduction equation.

H. Polachek (Washington, D. C.).

Brunk, H. D. Approximate solution of an initial value problem by generalized cardinal series. Quart. Appl. Math. **11**, 285–294 (1953).

Let $U(x, y)$ denote the temperature at point x and time $y\geqq -a$, $a>0$ on an infinite insulated rod. Set $f(x)=U(x, 0)$. The cardinal series

$$\sum_{k=-\infty}^{+\infty} f(k\lambda)\Phi(x\lambda^{-1}-k, y\lambda^{-2}),$$

$$\Phi(x, y)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-itx-yt^2)dt$$

which interpolates to $f(x)$ at the points $k\lambda$ ($k=\cdots, -2, -1, 0, 1, \cdots$) is set up. The author proves that under certain conditions the cardinal series converges to a function $f_\lambda(x, y)$ such that $f_\lambda(x, y)\to U(x, y)$, $y>-a$. Sufficient conditions for this to be the case are discussed in detail. A

probabilistic interpretation of this representation is also given. *P. Davis* (Washington, D. C.).

*Warschawski, S. E. **On the effective determination of conformal maps.** Contributions to the theory of Riemann surfaces, pp. 177–188. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The function which conformally maps a simply connected region on the interior of a circle can be found by solving the linear integral equation of Gerschgorin [Mat. Sbornik 40, 48–58 (1933)]. The convergence of the solution of this equation by iteration is considered and estimates of the error of the nth approximation are stated in terms of the second eigenvalue of the kernel. Bounds for the reciprocal of this eigenvalue are given and special estimates are given for nearly circular and nearly convex domains. The proofs of the theorems are to appear elsewhere. *C. Saltzer.*

*Birkhoff, G., Young, D. M., and Zarantonello, E. H. **Numerical methods in conformal mapping.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 117–140. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

The numerical approximation of the conformal mapping function which maps the interior or exterior of a smooth curve on a circle by digital machine methods is considered. Modifications of the integral equations of Theodorsen [NACA Rep. no. 411 (1931)] and of Gerschgorin [Mat. Sbornik 40, 48–58 (1933)] to which the conformal mapping problem can be reduced and of the difference equations to be used for computations in place of the integral equations are proposed. A new integral equation method for the solution of the "free boundary" problem is given. In addition several other methods are reviewed, and the convergence and computational difficulty of the various methods are evaluated both analytically and by application to a numerical example. *C. Saltzer* (Cleveland, Ohio).

Dobrovol'skiĭ, V. V. The general theory of mechanisms for the generation of plane curves. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 9, no. 36, 11–77 (1950). (Russian)

The constructions of projective geometry are implemented in mechanical devices to draw curves. A point constrained to a line is replaced by a pin constrained in a straight slot. A line passing through a fixed point is replaced by a slotted bar pivoted about a fixed pin. The transfer of angular rotation from one line to another is accomplished by locking the opposite sides of a parallelogram linkage to the rotating lines. The demonstrations are quite detailed and elementary with very little use of algebra. Some of the mechanisms have more theoretical than practical value.

A conic section can be drawn by a mechanism which simulates Pascal's theorem. If five points A, B, C, D, E of the conic are given, the variable sixth point M traces the conic. Fixed slots through BE and EC are used to guide pins which move, also, in three rotating slots which are hinged at H (the intersection of AB and CD) and points A and D. If the pin L lies on the triplet of lines (BE, HL, DL) and the pin K on (EC, HL, AK), then the pin M will be the intersection of AK and DL. Variations of the mechanism are shown corresponding to special cases of the theorem, namely, when any five elements are given (points and tangents).

Third and fourth degree curves are generated by compounding the simpler quadratic linkages, or by the use of four-bar and crosshead mechanisms. Mechanisms for quadratic transformations and Cremona transformations are described. *M. Goldberg* (Washington, D. C.).

*Booth, Andrew D., and Booth, Kathleen H. V. **Automatic digital calculators.** Academic Press Inc., New York; Butterworths Scientific Publications, London, 1953. 231 pp. (1 plate). $6.00.

This book is a good guide to the entire field of general purpose digital computers, covering both design and use aspects of them. On the design side, descriptions of the logical components carried as far as the usual block diagrams, but stopping short of the actual component values, give the reader a good idea of what a digital computer is and how it works. On the use side, the authors carry the reader through a number of examples of increasing complexity, and end the book with the topic of programming computers to play various games. *R. Hamming.*

*Freudenthal, H. **Machines pensantes.** Université de Paris, Paris, 1953. 16 pp.

Leondes, Cornelius, and Rubinoff, Morris. DINA, a digital analyzer for Laplace, Poisson, diffusion, and wave equations. Trans. Amer. Inst. Elec. Engrs. Part I. 71, 303–308; discussion, 308–309 (1952).

Brandon, P. S. Electronic analogue integration and differentiation. Electronic Engrg. 25, 476–477 (1953).

Gregory, Robert T. Computing eigenvalues and eigenvectors of a symmetric matrix on the ILLIAC. Math. Tables and Other Aids to Computation 7, 215–220 (1953).

Kartha, Gopinath. Double Fourier-synthesis. Punched card method. J. Indian Inst. Sci. 35, 332–338 (1953).

*Deschamps, G. A. **A hyperbolic protractor for microwave impedance measurements and other purposes.** Federal Telecommunication Laboratories, Nutley, N. J., 1953. 44 pp. $2.50.

Spence, Homer W. Systematization of tube surveillance in large-scale computers. Trans. Amer. Inst. Elec. Engrs. 70, 516–519 (1951).

Snyder, R. L., Jr. The input-output system of the EDVAC. Trans. Amer. Inst. Elec. Engrs. 70, 507–509 (1951).

Liénard, P. Représentation graphique de la résonance d'un oscillateur, utilisable pour l'étude de systèmes non linéaires. Acustica 3, 212–223 (1953).

Ratz, A. G., and Smith, V. G. A method of gating for parallel computers. Trans. Amer. Inst. Elec. Engrs. 70, 510–516 (1951).

Andersen, Einar. Practical formulas for accurate calculation by relative long distances of geographical coordinates or distances and azimuths on the international ellipsoid of rotation. Mém. Inst. Géodésique Danemark [Geodaetisk Instituts Skr.] (3) 16, 40 pp. (1953).

The resolution of a geodetic triangle for large arcs and the computation of distances along an extended geodesic is reviewed. An improved computational procedure suitable

for machine calculation and capable of high accuracy is presented. The method involves essentially a series solution in terms of the second eccentricity of the ellipsoid. The theory and representative numerical calculations are given in detail. *N. A. Hall* (Minneapolis, Minn.).

van der Hoek, U. H. A practical method for the calculation of a_x at various rates of interest; with numerical data based on the Dutch mortality table G.B.M. 1947–1949 (Makeham graduation). Verzekerings-Arch. Actuarieel Bijvoegsel **30**, 65*–75* (1953).

MECHANICS

Bereis, R. Über die symmetrische Rollung. Österreich. Ing.-Arch. **7**, 243–246 (1953).

Several theorems are derived regarding the motion of a curve rolling over an identical curve and symmetrically located with respect to their common tangent. For example, the path of a moving point is the anticaustic of its corresponding fixed point reflected from the fixed curve. The osculating circle of the centrode is always four times as large as the corresponding return circle. The focus of the osculating parabola through the point of contact is the Ball point of the moving centrode. The foci of the osculating conics coincide with the Burmester points (points whose paths have fourth order contact with their osculating circles). *M. Goldberg* (Washington, D. C.).

Matthieu, P. Über die Berechnung der Hypoidgetriebe. II. Ing.-Arch. **21**, 287–291 (1953).

The general polar coordinate equations of the tooth surface of the hypoid gear mating with the tooth surface of a given gear, which were reported in the author's earlier paper [Ing.-Arch. **21**, 55–62 (1953); these Rev. **14**, 808] for orthogonal axes, are now extended for skew axes. To facilitate computation, the set of coefficients in the original equations are replaced by a modified set. *M. Goldberg.*

Bat', M. I. On the determination of the solutions for the forces of operating linkage mechanisms depending upon the velocities and positions of the given links. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **9**, no. 34, 53–66 (1950). (Russian)

The general differential equations, similar to those used in electrical circuits, are derived giving the relations between the masses, moments of inertia, positions and forces of periodic mechanisms. These are applied to a Scotch yoke mechanism. Numerical coefficients in an expansion solution are evaluated. *M. Goldberg* (Washington, D. C.).

Abramov, B. M. Dynamical investigation of mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **10**, no. 41, 36–42 (1951). (Russian)

Skvorcova, N. A. Internal involute gearing with tooth-number difference of one. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **7**, no. 25, 85–90 (1949). (Russian)

If z and z' are the numbers of teeth, and $z-z'=1$, the transmission ratio is z. The feasibility of large z for internal involute gearing has been questioned because standard hob profiles cannot be used. Using the hob profile of Gavrilenko [A geometric theory of involute gearing, Mašgiz, 1949] the authoress has successfully produced gears of $z=24$, 31, 40, 49, and 100. *A. W. Wundheiler* (Chicago, Ill.).

***Pars, L. A. Introduction to dynamics.** Cambridge, at the University Press, 1953. xxii+501 pp. $6.00.

This textbook is intended for students who have a working knowledge of elementary calculus and plane statics. Vectors are introduced at the outset but coordinates are also used extensively. The content is confined to one- and two-dimensional motion of particles, rigid bodies and systems—without Lagrange's equations. Within this area the development is extremely thorough. Rectilinear motion is covered in great detail, including, e.g., libration motion, impulses, various resistance laws, fields of force and variable mass. The section on plane motion of particles includes, e.g., oscillators, the pendulum, brachistochrone, general constrained motion, projectiles, central orbits (treated very extensively), nonconservative forces, collisions. The plane motion of a rigid body includes, e.g., the motion of a lamina, d'Alembert's principle, centrodes, internal stresses, impulses. The plane motion of a system is carried as far as is reasonable without Lagrange's equations, including a detailed study of particular systems. A chapter on Dimensions concludes the text. The presentation is clear, the typography and format are admirable and there are hundreds of excellent problems and examples. *H. D. Block.*

Novoselov, V. S. Application of the methods of analytic mechanics to the computation of statically indeterminate systems. Vestnik Leningrad. Univ. **1952**, no. 2, 24–41 (1952). (Russian)

After an extended review of the method of Frank [Monatsh. Math. Phys. **23**, 225–239 (1912)] and Krutkow's modification [Doklady Akad. Nauk SSSR (N.S.) **11**, 213–215 (1936)], the author points out vagueness in the hypothesis relating to the potential of the reaction forces, and in the physical meaning of rigidity, among other difficulties, and discusses these points at some length. Much of the paper is concerned with detailed study of several illustrative examples. *R. E. Gaskell* (Seattle, Wash.).

de Broglie, Louis. Sur un théorème de M. Poincelot. C. R. Acad. Sci. Paris **237**, 586–588 (1953).

Extension to dynamical systems of the result obtained by Poincelot [same C. R. **237**, 382–384 (1953); these Rev. **15**, 271] for optical rays. [The extension is subject to the same criticism as that made in the review of the paper of Poincelot.] *J. L. Synge* (Dublin).

Cenov, I. On integral variational principles of analytic dynamics. Doklady Akad. Nauk SSSR (N.S.) **89**, 623–626 (1953). (Russian)

In the first section of this note, the author continues [see same Doklady (N.S.) **89**, 21–24, 225–228 (1953); these Rev. **14**, 916] his rephrasing of the equations of motion in terms of $\dot{T}$ (T=kinetic energy) expressed in terms of the generalized coordinates q and their time derivatives up to order three. The principle of least action is rederived in these terms for the holonomic case. The second part of the paper presents a form of Hamilton's principle for nonholonomic constraints. Here, apparently using the principle of humility, he assumes the commutatitivity relation $d\delta q=\delta dq$, on the grounds that some well-known men did so before him [see Dobronravov, Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 760–764 (1952); these Rev. **14**, 593]. This,

the reviewer believes, establishes the principle only for variations which limit the competing motions to (holonomic) subspaces, if any, containing the true motion and respecting the nonholonomic constraints.

A. W. Wundheiler (Chicago, Ill.).

Zagar, Francesco. Su un caso particolare del problema di più corpi e sue conseguenze astronomiche. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 610–618 (1951).

A qualitative discussion is given of the motion of n bodies subject to mutual attraction proportional to distance, of the motion of two bodies subject to Newtonian interaction but perturbed by a third body attracting or repelling both with a force proportional to distance, and of the motion of a particle which is attracted by a homogeneous solid, the potential being obtained from a force proportional to distance. *W. Kaplan* (Ann Arbor, Mich.).

Hahn, Wolfgang. Die mechanische Deutung einer geometrischen Differenzgleichung. Z. Angew. Math. Mech. **33**, 270–272 (1953).

A weightless elastic cable carries an infinite number of concentrated masses. If both the weights and mutual distances of successive masses decrease in geometric progressions with the same ratio q, the author shows that the equations of motion (under gravity) of the system leads to a geometric difference ($=q$-difference) equation which can be solved in terms of Heine's basic hypergeometric series.

A. Erdélyi (Pasadena, Calif.).

Hydrodynamics, Aerodynamics, Acoustics

***Truesdell, C. Vorticity and the thermodynamic state in a gas flow.** Mémor. Sci. Math., no. 119. Gauthier-Villars, Paris, 1952. 53+i pp. 900 francs.

Cet exposé est consacré à l'étude des propriétés des mouvements d'un fluide parfait compressible; la nécessité de tenir compte de l'équation d'état fait qu'une telle étude doit utiliser à la fois la mécanique et la thermodynamique; elle n'avait point jusqu'à présent fait l'objet d'un exposé d'ensemble systématique et suffisamment rigoureux. L'objectif précis du mémoire est ainsi défini: "to characterize irrotational gas flows in thermodynamical terms, and in rotational gas flows to search out the relations which bind the vorticity to the thermodynamic variables."

L'auteur commence par établir un certain nombre de résultats purement cinématiques; ensuite il passe en revue les propriétés dynamiques que l'on peut établir sans faire appel à la thermodynamique (les théorèmes dits de Bernoulli étendus aux mouvements de Beltrami, etc.); enfin dans le reste du mémoire l'équation d'état et les grandeurs thermodynamiques sont utilisées et de nombreux théorèmes répondant à l'objectif défini ci-dessus sont exposés. L'auteur étudie en particulier les gaz de Prim (gaz pour lesquels l'équation d'état exprime la densité comme produit d'une fonction de la pression par une fonction de l'entropie spécifique); il donne pour ces gaz le principe de substitution [M. Munk et R. C. Prim, Proc. Nat. Acad. Sci. U. S. A. **33**, 137–141 (1947)] et les équations canoniques [M. Munk et R. C. Prim, loc. cit.; B. Hicks, P. Guenther et R. Wasserman, Quart. Appl. Math. **5**, 357–361 (1947); ces Rev. **9**, 112].

Le théorème 23 est justiciable de certaines critiques et d'autres critiques mineures peuvent être faites. Mais cela n'enlève rien à la valeur du mémoire où un effort méritoire et réussi a été fait pour préciser les hypothèses nécessitées par chaque théorème, pour rendre les démonstrations rigoureuses et l'enchaînement des théorèmes logique et naturel. Le mémoire se présente comme un instrument de travail extrêmement précieux. *R. Berker* (Istanbul).

Slezkin, N. A. The fundamental equations of the motion of a deformable medium of particles with variable mass. Doklady Akad. Nauk SSSR (N.S.) **79**, 33–36 (1951). (Russian)

Slezkin, N. A. The differential equations of the deformation process. Doklady Akad. Nauk SSSR (N.S.) **80**, 561–564 (1951). (Russian)

***Heins, Albert E. On gravity waves.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 75–86. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

L'auteur considère le problème de la propagation des ondes de gravité en profondeur finie, en présence d'un obstacle constitué par un demi-plan horizontal immergé au-dessous de la surface libre, lorsque la direction de propagation fait un angle quelconque avec le bord de l'obstacle. Les équations étant linéarisées cela conduit à résoudre pour l'équation

$$\phi_{xx}+\phi_{yy}-k^2\phi=0$$

un problème aux limites, les conditions à la frontière étant de la forme $\phi_y=\beta\phi$ ($\beta>0$) sur la ligne libre au repos, et $\phi_y=0$ sur le fond et sur l'obstacle. Par l'introduction de fonctions de Green, ce problème mixte est ramené de deux manières différentes à la résolution d'une équation intégrale de Wiener-Hopf. L'allure asymptotique de la solution est étudiée et en particulier la valeur de l'amplitude de l'onde réfléchie et de l'onde transmise. *R. Gerber* (Toulon).

Ursell, F. The long-wave paradox in the theory of gravity waves. Proc. Cambridge Philos. Soc. **49**, 685–694 (1953).

La propagation des ondes de gravité de grande longueur d'onde (par rapport à la profondeur) a donné lieu à plusieurs théories approchées, dont les résultats sont parfois contradictoires. Ainsi, selon la seconde approximation d'Airy une onde progressive, de hauteur non nulle, subit nécessairement un changement de forme, alors que d'autres approximations (Rayleigh, Boussinesq, ⋯) ont permis de construire des solutions approchées de l'onde solitaire.

L'auteur confronte et discute ces diverses théories pour chercher à en expliquer les divergences. Le rôle primordial du nombre sans dimensions $\eta_0\lambda^2/h^3$ (η_0 hauteur de l'onde, λ longueur d'onde, h profondeur) est mis en évidence. Suivant la valeur de ce nombre, c'est l'une ou l'autre des approximations qui représente le mieux le phénomène, le résultat d'Airy, par exemple, supposant que $\eta_0\lambda^2/h^3$ soit grand. Il est aussi montré que la théorie linéarisée est valable, non seulement lorsque η_0/λ est petit, mais encore à la condition que $\eta_0\lambda^2/h^3$ soit également petit. Le travail se termine par des remarques sur la génération de l'onde solitaire, et sur le déferlement de la houle sur une plage inclinée. Indiquons que l'auteur opère en variables de Lagrange. *R. Gerber* (Toulon).

Burns, J. C. Long waves in running water. With an appendix by M. J. Lighthill. Proc. Cambridge Philos. Soc. **49**, 695–706 (1953).

L'A. étudie l'influence d'une inégale répartition des vitesses sur la propagation des ondes de gravité, en eau courante peu profonde, dans un canal rectiligne et horizontal. Lorsque la vitesse est une fonction linéaire de la profondeur, une théorie non linéaire permet d'expliciter la vitesse de propagation de l'onde; mais pour des répartitions de la vitesse dans le courant non troublé plus générales il devient nécessaire de linéariser le problème en supposant que la perturbation est petite. On obtient alors une équation reliant la célérité de l'onde à la vitesse moyenne de l'écoulement, et cette équation est résolue dans des cas particuliers, dont celui d'une répartition parabolique des vitesses. Il est montré qu'en général la célérité absolue de l'onde a deux valeurs, l'une supérieure au maximum de la vitesse du courant, l'autre inférieure au minimum de cette vitesse.

R. Gerber (Toulon).

Garabedian, P. R. Oblique water entry of a wedge. Comm. Pure Appl. Math. **6**, 157–165 (1953).

This paper adds another to the relatively rare examples of exact solutions of unsteady free boundary problems. The author derives explicit formulas for a one-parameter family of flows, each describing in an appropriate sense the oblique water entry of a wedge. These flows have the property that the constant free surface pressure has unequal values on opposite sides of the wedge. The problem is reduced to a steady one by the similarity requirement that the velocity fields be similar at all times. The flows of H. Wagner [Z. Angew. Math. Mech. **12**, 193–215 (1932)], which describe the vertical entry of a symmetrical wedge, are not included among the author's solutions, and have not as yet been given in closed form. The details are based on a technique of analytic continuation previously applied by the author and D. Spencer [J. Rational Mech. Anal. **1**, 359–409 (1952); these Rev. **14**, 102] and by H. Lewy [Comm. Pure Appl. Math. **5**, 413–414 (1952); these Rev. **14**, 594] to various free boundary problems. *D. Gilbarg* (Bloomington, Ind.).

Tulin, M. P. Steady two-dimensional cavity flows about slender bodies. The David W. Taylor Model Basin, Washington, D. C., Report **834**, iv+26 pp. (1953).

This paper is an approach to the problem of plane cavitational flow by way of the linearized theory. The linearization is the same as that in thin airfoil theory and leads to a simple integral equation of known type for which the solution can be given explicitly. The usefulness of the method for obtaining quantitative results is apparent in that the calculation of cavity shapes and drags for arbitrary slender bodies at any cavitation number is reduced to one of quadratures, whereas such calculations in the exact theory are extremely formidable in general. The validity of the method is supported by calculations on a family of wedge profiles in which the author compares results of the linearized and exact theories and finds close agreement at small angles. *D. Gilbarg* (Bloomington, Ind.).

Schultze, Ernst. Zur Berechnung der Druckpunktverteilung über die Spannweite für Flügel mit kleinem Seitenverhältnis. Z. Angew. Math. Physik **4**, 207–214 (1953).

After pointing out the relation of the success of Weissinger's integro-differential equation for the spanwise variation of lift on wings of finite span to a theorem of two-dimensional thin-wing theory for circular arc profiles, the author obtains the following extension of this theorem of the two-dimensional theory. Any profile consisting of a flat plate part, a circular arc part, and an S-curve part is equivalent, in regard to lift and moment, to two concentrated vortices at distances $x_1=.0955c$ and $x_2=.6545c$ from the leading edge, provided the tangential flow condition is satisfied at two points possessing distances $.3455c$ and $.9045c$ from the leading edge (c=wing chord). The author suggests that this theorem may be utilized for the derivation of two simultaneous integro-differential equations for spanwise variation of lift and moment. [The results which would be obtained in this manner have been given by the reviewer in Proc. Nat. Acad. Sci. U. S. A. **35**, 208–215 (1949); these Rev. **10**, 753.] *E. Reissner* (Cambridge, Mass.).

Imai, Isao. Discontinuous potential flow as the limiting form of the viscous flow for vanishing viscosity. J. Phys. Soc. Japan **8**, 399–402 (1953).

A qualitative discussion in support of the thesis that "if the continuous potential flow around a given obstacle has such a pressure distribution that the boundary-layer separation does not occur, then the said potential flow (plus the boundary layer) is the possible asymptotic flow pattern for vanishing viscosity. Otherwise, the asymptotic flow is given by the discontinuous flow of the Kirchhoff type, where the separation points of the free streamlines are coincident with those predicted by the boundary layer theory from the pressure distribution given by the same discontinuous flow. In this case, there are two possibilities: one is that the free streamlines have a finite curvature equal to that of the obstacle at the separation point and the other is that the separation occurs at a corner (if it exists) of the obstacle (just as in the case of a flat plate)." *D. Gilbarg.*

Kawaguti, Mitutesi. Discontinuous flow past a circular cylinder. J. Phys. Soc. Japan **8**, 403–406 (1953).

The author tests Imai's conclusion in the paper reviewed above by approximate calculations to determine the points of boundary-layer separation corresponding to the family of discontinuous (Kirchhoff) potential flows past circular-arc profiles. He finds that the points of boundary-layer separation and of free-streamline detachment coincide (approximately) when the free-streamline curvature at detachment equals that of the circular arc. This is in agreement with Imai's assertion, although at variance with experiment.

D. Gilbarg (Bloomington, Ind.).

Westervelt, Peter J. Hydrodynamic flow and Oseen's approximation. J. Acoust. Soc. Amer. **25**, 951–953 (1953).

It is proved that a linearization of the hydrodynamic equations in the manner of Oseen does not lead to a description of the steady phenomena which occur in connection with oscillatory flow past solid obstacles.

From the author's summary.

Hasimoto, Hidenori. On the flow of a viscous fluid past an inclined elliptic cylinder at small Reynolds numbers. J. Phys. Soc. Japan **8**, 653–661 (1953); errata, **9**, 146 (1954).

This paper concerns viscous flow past an inclined elliptic cylinder at small Reynolds number. By Oseen's approximation, solutions in terms of elliptic coordinates are obtained. Lift and drag coefficients are expanded in powers of Reynolds number but the evaluation of the coefficients of the series is carried only to the second order of Reynolds number.

Y. H. Kuo (Pasadena, Calif.).

*McDowell, E. L. **Axial oscillations of a viscous fluid.** Proceedings of the Third Midwestern Conference on Fluid Mechanics, University of Minnesota, March 23, 24, and 25, 1953, pp. 631–634. University of Minnesota, Minneapolis, Minn., 1953. $6.00.

This paper obtains a solution of the Navier-Stokes equations for the motion produced by an infinite circular cylinder performing harmonic oscillations parallel to its axis.

Y. H. Kuo (Pasadena, Calif.).

Thomas, L. H. The stability of plane Poiseuille flow. Physical Rev. (2) **91**, 780–783 (1953).

The characteristic value problem arising from the study of the stability of the plane Poiseuille flow is solved by machine computation. The fourth-order differential equation is replaced by a difference system of the same order with a truncation error involving the eighth derivative, and the resulting system of linear algebraic equations is solved by direct Gaussian elimination, which avoids the difficulties due to rapid exponential growth of error. The numerical results found agree closely with those obtained by this reviewer by asymptotic series; but the present calculations contain also information about some amplified and damped disturbances. The solution function is also given in the present calculation. [Reviewer's remark: Recently, S. F. Shen computed the amplification by analytical methods and found results in agreement with the present calculation.]

C. C. Lin (Cambridge, Mass.).

Corrsin, Stanley. Interpretation of viscous terms in the turbulent energy equation. J. Aeronaut. Sci. **20**, 853–854 (1953).

Iglisch, Rudolf. Elementarer Existenzbeweis für die Strömung in der laminaren Grenzschicht zur Potentialströmung $U=u_1x^m$ mit $m>0$ bei Absaugen und Ausblasen. Z. Angew. Math. Mech. **33**, 143–147 (1953).

The author proves by elementary methods the existence of a solution of the boundary layer equation

$$f'''+ff''+\beta(1-f'^2)=0$$

under the boundary conditions $f(0)=C, f'(0)=0, f'(\infty)=1$, and satisfying the side condition $f'(\eta)>0$, $0<\eta<\infty$; the constant C may be arbitrary while $\beta>0$. The function f describes in the customary manner the flow in a laminar boundary layer, which in this case is along an infinite wedge of angle $\beta\pi$. The significance of the constant C is that it determines a certain distribution of normal velocities along the wall (tangential component $=0$), $C=0$ corresponding to the usual condition of adherence. The existence problem for $\beta\geq 0$ and $C=0$ was treated fully by H. Weyl [Ann. of Math. (2) **43**, 381–407 (1942); these Rev. **3**, 284] by the fixed-point method. The results of the present paper on the other hand are based entirely on elementary arguments issuing directly from the differential equation. The case $\beta=0$ is not included, since for this value of β, as the author and D. Grohne have shown [Veröffentl. Math. Inst. Techn. Hochsch. Braunschweig, Bericht 1/45 (1945)], a solution (with $f'>0$) exists only for $C\geq -1.2385\cdots$.

D. Gilbarg (Bloomington, Ind.).

Tan, H. S. On laminar boundary layer over a rotating blade. J. Aeronaut. Sci. **20**, 780–781 (1953).

The author proposes to extend to smaller radii Fogarty's calculations [same J. **18**, 247–252 (1951); these Rev. **14**, 328] of the boundary layer on a cylindrical blade rotating about an axis normal to the cylindrical axis. The differential equations for the nth approximation are found, where Fogarty's corresponds to $n=1$. The special case of a rotating flat plate is partly carried through, but complete numerical results are not given here.

W. R. Sears (Ithaca, N. Y.).

Merk, H. J., and Prins, J. A. Thermal convection in laminary boundary layers. I. Appl. Sci. Research A. **4**, 11–24 (1953).

Consideration is given to configurations where the partial differential equations for a laminar free convection boundary layer may be reduced to ordinary differential equations. Conditions presented as sufficient rely on similarity of velocity and temperature profiles along the body. The resulting differential equations are:

$$f'''+Bff''-2f'^2+\chi=0,$$
$$\chi''+B\,\mathrm{Pr}\,f\chi'=0,$$

with the boundary conditions

$$f=0,\quad f'=0,\quad \chi=1 \quad \text{for} \quad \eta=0$$
$$f'=0,\quad \chi=0 \quad \text{for} \quad \eta=\infty$$

where f determines the velocity stream function, χ the temperature profile, and B is a constant dependent on the surface configuration. Approximate polynomial solutions indicate for the cases of a vertical flat plate, the stagnation point on a circular cylinder and sphere, and vertical cylinder and cone, that the heat transfer coefficient obtained is consistent with the general formulation $\mathrm{Nu}=C(\mathrm{Gr}\,\mathrm{Pr})^{\frac{1}{4}}$ obtained by other investigators.

N. A. Hall.

Schenk, J., and Dumoré, J. M. Heat transfer in laminar flow through cylindrical tubes. Appl. Sci. Research A. **4**, 39–51 (1953).

The previous work of the authors [van der Does de Bye and Schenk, Appl. Sci. Research A. **3**, 308–316 (1952); these Rev. **14**, 332] is extended to circular tubes where again the effect of finite transmissivity of the walls is included. In addition to providing tabular data on solutions of the flow equations for a series of values of the wall Nüsselt number, the association of the functions arising in this and the previous paper with the Whittaker confluent hypergeometric functions is established. Certain general numerical results as to zeros and eigenfunctions for this class of functions are given.

N. A. Hall (Minneapolis, Minn.).

Bugaenko, G. A. On free heat convection in vertical cylinders of arbitrary cross-section. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 496–500 (1953). (Russian)

Assuming vertical cylindrical surfaces with similar temperature profiles in vertical direction, the laminar free-convection equations are formulated and solved for rectangular and circular cross-sections but with arbitrary boundary-temperature distribution.

N. A. Hall.

Blanch, Gertrude, and Fettis, Henry E. Subsonic oscillatory aerodynamic coefficients computed by the method of Reissner and Haskind. J. Aeronaut. Sci. **20**, 851–853 (1953).

Marchetti, Luigi. Determinazione del campo di velocità di una corrente subsonica piana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 42–46 (1953).

The author sets himself the problem: given the streamlines of the steady two-dimensional homentropic flow due to a cylindrical obstacle in a uniform stream, find the velocity distribution. The solution is, of course, well known:

the product of density and velocity on a given streamline varies in inverse proportion to its distance from an adjacent one, and the density can be eliminated by use of the universal relationship between density and velocity provided by Bernoulli's equation. The author gets a result equivalent to this, but quite obscured by a cumbersome notation and a proof of fantastic length. *M. J. Lighthill.*

†**Bergman, Stefan. Methods for the determination of subsonic flows around profiles.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 705–713. The American Society of Mechanical Engineers, New York, N. Y., 1952.

The author describes briefly (with references to other publications) how using the kernel-function and integral operators one could construct subsonic flow patterns. He considers the case of flows bounded by straight segments and free streamlines, and also discusses the singularities in the hodograph plane due to the multiply covered character of the hodograph. No examples are given. *L. Bers.*

Cole, J. D. Note on nonstationary slender-body theory. J. Aeronaut. Sci. **20**, 798–799 (1953).

Cole, Julian D. Drag of a finite wedge at high subsonic speeds. J. Math. Physics **30**, 79–92 (1951).

Cet important article est consacré au calcul de la trainée d'un obstacle en forme de coin, lorsque le nombre de Mach à l'infini amont est très légèrement inférieur à l'unité. La méthode utilisée est celle de l'hodographe. En fait l'auteur simplifie les équations en supposant l'angle du coin suffisamment petit, ce qui conduit à l'approximation de Tricomi pour laquelle les formules de similitude de von Kármán sont strictement valables. Une analyse des conditions aux limites montre que, dans le plan de l'hodographe, la singularité correspondant à l'écoulement donné est un doublet; la discussion laisse prévoir de plus que des simplifications supplémentaires sont permises si bien qu'en définitive le problème se réduit à un problème de Dirichlet dans un domaine intérieur à un contour dont une partie appartient à la ligne sonique. C'est cette simplification qui permet l'élégante solution qui nous est proposée; du point de vue physique cette hypothèse revient à admettre que la ligne sonique qui part de l'épaulement de l'obstacle est une ligne droite perpendiculair à la direction de l'écoulement donné. La solution s'exprime par une intégrale de Laplace portant sur un produit de fonctions de Bessel d'ordre 1/3. Le coefficient de trainée s'exprime à l'aide d'un développement en série (en réalité, seule la contribution due à la partie avant de l'obstacle est calculée), et la comparaison du résultat théorique obtenu avec divers résultats expérimentaux montre une concordance très satisfaisante. *P. Germain.*

Trilling, Leon. Transonic flow past a wedge at zero angle of attack. Z. Angew. Math. Physik **4**, 358–375 (1953).

Cette étude est consacrée à l'étude de l'écoulement transsonique autour d'un losange pour des nombres de Mach à l'infini légèrement inférieurs à l'unité. Comme dans l'article de J. D. Cole [voir la précedente analyse], l'équation de Tricomi et l'approximation correspondante sont utilisées. Mais le problème actuel est beaucoup plus difficule, car il est théoriquement impossible d'envisager une région correspondant à l'écoulement subsonique et transsonique, qui ne soit pas influencée par l'écoulement proprement supersonique et les ondes de choc qui s'y trouvent nécessairement. L'auteur commence donc par discuter les conditions aux limites et les hypothèses qui peuvent être faites en vue d'arriver à une formulation mathématique du problème; en conclusion le schéma retenu par l'auteur conduit à la solution d'un problème de Tricomi (avec une singularité d'un type connu dans le demi plan elliptique) et d'un problème purement hyperbolique. Ce dernier (dont les données semblent surabondantes à l'auteur de cette analyse) est résolu par la méthode des caractéristiques; l'essentiel de l'article est consacré à la solution effective du problème de Tricomi. Celle ci est atteinte par étapes, il faut d'abord construire les solutions donnant le type de singularité voulu et ayant le comportement convenable au voisinage du bord d'attaque, utiliser la méthode des images pour satisfaire une partie des conditions aux limites dans le demi plan elliptique; les conditions le long de la caractéristique sont approximativement réalisées en utilisant une technique due à Guderley et Yoshihara [voir la deuxième analyse ci-dessous]; enfin il est nécessaire d'ajouter encore certaines solutions pour satisfaire à toutes les conditions données.

Les principales caractéristiques de l'écoulement sont calculées numériquement à partir de la solution précédente; la position du choc normal qui termine la région supersonique s'obtient en introduisant dans la courbe représentant les variations de la vitesse le long de la corde, la discontinuité nécessaire pour obtenir une valeur unique de la vitesse en chaque point. Les résultats concernant la trainée sont en bon accord avec les résultats expérimentaux connus. *P. Germain* (Providence, R. I.).

Trilling, L., and Walker, K., Jr. On the transonic flow past a finite wedge. J. Math. Physics **32**, 72–79 (1953).

Ce mémoire apporte un perfectionnement à l'article de J. D. Cole consacré à ce problème [voir la deuxième analyse ci-dessus]. Si on utilise une approximation classique dans l'étude des écoulements transsoniques, les coordonnées x et y du plan de l'écoulement sont des fonctions des composantes u et v des vitesses de perturbation satisfaisant à: $uy_u - x_u = 0$, $y_v - x_v = 0$ (u et v sont proportionnelles aux composantes ordinaires des vitesses de perturbation par rapport à la vitesse critique, $u = M^2 - 1$ à l'infini amont). La fonction $y(u, v)$ est alors solution d'une équation de Tricomi et dans la région elliptique ($u < 0$) satisfait aux conditions aux limites: $y(u, v_0) = 0$, $y(u, 0) = 0$, elle admet au point $u = M^2 - 1$, $v = 0$, un doublet comme singularité (v_0 est proportionnelle à l'angle de la paroi avec Ox). Pour simplifier, J. D. Cole admettait que la région subsonique était déterminée de façon suffisante en supposant que la ligne sonique issue de l'épaulement était parallèle à l'axe des y, et résolvait explicitement le problème ainsi posé. Les auteurs reprennent le travail de J. D. Cole mais déterminent la solution en écrivant la condition fournie par la détente de Prandtl-Meyer qui prend naissance à l'épaulement. Il faut alors résoudre un problème de Tricomi; cette résolution s'effectue à l'aide d'un développement en série de Fourier, les données sur la caractéristique étant remplacées par la condition intégrale de Tricomi sur la ligne sonique ($u = 0$). Les coefficients du développement satisfont à une infinité d'équations linéaires et les 35 premiers sont calculés numériquement. Les formes trouvées pour la ligne sonique et les valeurs du coefficient de trainée sont comparées avec les résultats expérimentaux. Il faut signaler que ce schéma, bien qu'apportant une amélioration à celui envisagé par J. D. Cole, ne rend pas compte encore de la description exacte du problème physique, en particulier de l'onde de choc.

P. Germain (Providence, R. I.).

Guderley, G., and Yoshihara, H. The flow over a wedge profile at Mach number 1. J. Aeronaut. Sci. **17**, 723–735 (1950).

Dans cet article, aujourd'hui classique, les auteurs étudient l'écoulement transsonique autour d'un losange lorsque la vitesse à l'infini amont est sonique. L'écoulement proprement transsonique se termine à la frontière transsonique (limiting Mach wave); il est étudié par la méthode de l'hodographe en utilisant l'approximation de Tricomi. En fait, du point de vue mathématique, le problème se réduit à un problème de Tricomi une fois que l'on a introduit dans la solution la singularité convenable correspondant à l'écoulement à l'infini amont dans le plan physique. La solution complète du problème est obtenue en faisant appel à une succession de procédés ingénieux susceptible de favoriser le calcul numérique: méthode des images, modification de l'énoncé initial afin de profiter au maximum des solutions périodiques, utilisation d'une analogie formelle avec un problème de fluides incompressibles pour résoudre une équation intégrale. La partie proprement supersonique de l'écoulement est calculée pour une forme spéciale de la partie terminale du profil, choisie pour absorber les ondes de compression provenant de la ligne sonique. La trainée du profil ainsi formé—qui diffère très peu d'un losange—est calculée en fonction de l'épaisseur relative. *P. Germain.*

Meksyn, D. Integration of the equations of transonic flow in two dimensions. Proc. Roy. Soc. London. Ser. A. **220**, 239–254 (1953).

Il s'agit dans ce travail d'une méthode approchée pour calculer l'écoulement autour d'un profil lorsque le nombre de Mach à l'infini amont est inférieur à l'unité. L'étude est faite dans le plan physique en utilisant comme système de coordonnées curvilignes les équipotentielles $\alpha=$ const. et les lignes de courant $\beta=$const. L'équation du potentiel des vitesses est une équation de Poisson dans laquelle la répartition de masses dépend elle même de la fonction inconnue; c'est dans ce dernier terme (qui dépend du nombre de Mach et qui est nul pour l'écoulement incompressible) que sont faites les approximations. Tout d'abord les dérivées par rapport à β sont négligées; ensuite la dérivée première en α du potentiel est considérée comme constante dans des intégrations et dérivations. Ceci permet l'intégration classique de l'équation de Poisson bien que le résultat soit assez curieux puisqu'il s'exprime en fonction de la dérivée première en α de la fonction inconnue. En écrivant l'équation de compatibilité qui en résulte on obtient une équation algébrique du 5ième degré dont la résolution fournit directement la quantité exprimant l'effet de la compressibilité. De la discussion relative aux racines de cette équation l'auteur pense pouvoir trouver une estimation de la valeur du nombre de Mach pour lequel un choc fait son apparition.

Les calculs sont développés dans deux cas particuliers, celui du cercle, et celui d'un profil déjà considéré par C. Kaplan [NACA Rep. no. **768** (1943); ces Rev. **9**, 477] et confrontés avec des résultats obtenus par d'autres méthodes; l'accord est très satisfaisant et c'est sans doute la meilleure justification d'une méthode qui comporte de nombreuses approximations. *P. Germain* (Providence, R. I.).

Vincenti, Walter G., and Wagoner, Cleo B. Transonic flow past a wedge profile with detached bow wave. NACA Rep. no. **1095**, ii+30 pp. (1952).

[This report supersedes NACA Tech. Notes no. **2339** (1951) and no. **2588** (1951).] The problem considered is the flow past a doubly symmetrical double-wedge aerofoil at zero incidence with a detached bow shock wave. Rotation is neglected and the problem examined in the first place in the hodograph plane, using small-disturbance theory (Tricomi's equation) and what is essentially the formulation of Guderley [Tech. Rep. no. **F-TR-2168-ND**, AAF, Air Materiel Command, Wright Field, 1947]. Difficulties with convergence in the hyperbolic region, which arise when relaxation techniques are applied to equations of mixed type, were avoided by replacing the boundary conditions in the supersonic zone by conditions on the sonic line; this was done by making use of a formula given by Tricomi [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **14**, 133–247 (1923) = Grad. Div. Appl. Math., Brown Univ., Translation no. **A9-T-26** (1948)], which relates the behaviour of the stream function on the sonic line to its behaviour on a characteristic. The boundary-value problems in the subsonic and supersonic parts of the flow field could thus be formulated separately. There is a detailed discussion (which may well be of use in other instances where detached shock waves occur) of the reduction of the boundary-value problem to a system of finite-difference equations and the application of relaxation methods to their solution. At all stages the work is considered in relation to previous work on flow past wedges. The chief results found are (i) that the subsonic region extends over a large vertical distance even when the detached shock wave is quite close to the leading edge, (ii) that as the free stream Mach number increases from 1, the local Mach number at a point on the surface of the profile also increases (but very slowly at first) and (iii) that the drag coefficient of the whole profile varies only slightly near sonic flight speed, decreases rapidly near the Mach number for attachment of the bow wave, and then decreases at a less and less rapid rate in the range of purely supersonic flow.

D. C. Pack (Glasgow).

Pack, D. C., and Pai, S. I. Similarity laws for supersonic flows. Quart. Appl. Math. **11**, 377–384 (1954).

The authors express in terms of dimensionless variables the linearized boundary condition and the partial differential equation, including some non-linear terms, for the perturbation potential of a steady irrotational flow at Mach number M over wings of thickness τ, aspect ratio A, and mean chord c. Boundary condition and linear terms of the equation yield as necessary conditions for a family of similar flows (i) $(M^2-1)\tau^{-2n}=K_1^2$, (ii) $A(M^2-1)^{\frac{1}{2}}=K_2$, (iii) $n+n'=1$, where K_1 and K_2 are constants, and $c\tau^{-n}$ and $c\tau^{n'}$ are scale factors for spanwise coordinate and perturbation potential. For M not too far from unity, invariance is imposed on the coefficient of the non-linear term most important for transonic flow to yield (iv) $M^2\tau^{n'-2n}=K_3^2$. A plausible argument for choosing $K_3=1$ and the relations above yield $n=n(M, K_1)$ with $\lim n=1/3$ as $M\to 1$ in agreement with the transonic similarity law. For large M invariance is imposed on the coefficient of another non-linear term to yield (v) $M^2\tau^{n'}=K_4^2$ instead of (iv). To assure $\lim n=-1$ as $M\to\infty$ as required by the hypersonic similarity law one must choose $K_4=K_1$, and again $n=n(M, K_1)$ for some $n(M, K_1)$. The generalized transonic-supersonic (supersonic-hypersonic) similarity law (i), (ii), (iii) and (iv) (or (v)) ceases to be valid for n too close to zero. Similarity laws for two-dimensional (axisymmetric) flows can be obtained by passing to the limit as $A\to\infty$ (0).

J. H. Giese.

*Carrier, G. F., and Yen, K. T. **On the construction of high-speed flows.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 55–60. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

It is shown how new two-dimensional flows of inviscid gas can be obtained by generalizing, in integral form, the solutions of the hodograph equation. Two examples are given. The reader is referred for definitions of the functions used to a paper which is difficult to obtain outside the U. S. A. The function $\omega(\tau)$ is, in fact, Lighthill's $s(\tau)$ and the other functions used are defined in Lighthill's work [Proc. Roy. Soc. London. Ser. A. **191**, 341–351 (1947); these Rev. **9**, 350; **11**, 870]. *D. C. Pack* (Glasgow).

Benedikt, Elliot T. Open-nose body of revolution of minimum drag at small supersonic speeds. J. Aeronaut. Sci. **20**, 720–722 (1953).

That profile of an open nose body of revolution, having prescribed nose and base radii, which minimizes the external drag is determined by applying variational calculus to Lighthill's slender body formula [Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2003 (1945); these Rev. **8**, 108]. The extremizing integral equation has the singular kernel $(x-y)^{-1}$ and is solved by an inversion formula due to Tricomi [Quart. J. Math., Oxford Ser. (2) **2**, 199–211 (1951); these Rev. **13**, 231]. (The reviewer sees no advantage of Tricomi's method over the older method of Hilbert in this case; indeed the solution to the integral equation in question is well known [W. Magnus and F. Oberhettinger, Formulas and theorems . . . , Chelsea, New York, 1949, p. 140; these Rev. **10**, 532].) The result is a generalization of the well known Kármán ogive [Convegno Sci. Fis. Mat. Nat., no. 5, 1935, Reale Accad. Italia, Roma, 1936, pp. 222–283], although this is not mentioned. *J. W. Miles* (Los Angeles, Calif.).

*Rosenbloom, Paul C. **The linearized theory of supersonic flow about a thin airfoil.** Proceedings of the Third Midwestern Conference on Fluid Mechanics, University of Minnesota, March 23, 24, and 25, 1953, pp. 319–334 University of Minnesota, Minneapolis, Minn., 1953. $6.00.

M. Riesz's method [Acta Math. **81**, 1–223 (1949); these Rev. **10**, 713] is applied to the problem of the title. The work (first carried out by the author in 1946) is now of only limited interest in comparison with the simpler procedure of Evvard [NACA Rep. no. 951 (1950); these Rev. **12**, 454] and the generalizations of Busemann's conical flow technique [e.g., S. Goldstein and G. N. Ward, Aeronaut. Quart. **2**, 39–84 (1950); these Rev. **15**, 177]. *J. W. Miles.*

Chester, W. Supersonic flow past wing-body combinations. Aeronaut. Quart. **4**, 287–314 (1953).

The author considers the mutual interference between a slender body and a thin wing whose leading edges are supersonic near the wing body junction (e.g., a wing without sweep-back). Operational methods are used to find an approximate solution based on the linearised equation of supersonic flow, but the errors relative to the exact equations of potential flow are also estimated. An ingenious procedure, which depends indirectly on slender-body theory, is employed to determine the flow in the region of the wing-body junction. Where necessary, the lift beyond the tips is cancelled by the use of a Green's function method which was introduced into supersonic flow by J. C. Gunn [Philos. Trans. Roy. Soc. London Ser. A. **240**, 327–373 (1947); these Rev. **9**, 313].

It is found, among other things, that if wing and body are of the same incidence and the aspect ratio of the wing is greater than $2(M^2-1)^{-1/2}$ then the lift is equivalent to that of the complete wing when isolated. If the body is an infinite cylinder of radius a then, with the same restriction on M, the drag of the wing is reduced by a factor $(1-2a/b)$ where $2b$ is the span of the wing.

This is an interesting contribution to the solution of a very intricate problem. The publication of more experimental evidence will be required for the definitive evaluation of the various methods which have been put forward to deal with this subject. *A. Robinson* (Toronto, Ont.).

Krzywoblocki, M. Z. V. Of the extension of Friedrichs' theorem on non-vanishing of the Jacobian to rotational inviscid isentropic and viscous flows. Bull. Soc. Math. Grèce **27**, 68–84 (1953).

Friedrichs' theorem on the non-occurrence of a limiting line in transsonic irrotational flows [Communications on Appl. Math. **1**, 287–301 (1948); these Rev. **10**, 638] is based on the proof that the Jacobian of the mapping from the hodograph to the physical plane is non-vanishing. The author of the present paper extends this result to include rotational isentropic flows of an inviscid fluid and also flows of a viscous fluid. *D. Gilbarg* (Bloomington, Ind.).

Bruniak, R. Zur Struktur des Verdichtungsstosses. Österreich. Ing.-Arch. **7**, 128–134 (1953).

A discussion consisting primarily of known results concerning the steady one-dimensional shock profile with viscosity and without thermal conductivity. *D. Gilbarg.*

Nyborg, Wesley L. Acoustic streaming equations: laws of rotational motion for fluid elements. J. Acoust. Soc. Amer. **25**, 938–944 (1953).

Since the time of Stokes the interpretation of the vorticity vector $\mathbf{R}$ as the angular velocity of a rigid spherical element has been familiar. Carrying this idea to a higher degree of approximation, the author obtains a new and interesting interpretation for the terms in the Euler-Helmholtz-Beltrami kinematical vorticity rate equation

$$(*) \qquad \frac{d\mathbf{R}}{dt}+(\nabla\cdot\mathbf{u})\mathbf{R}-(\mathbf{R}\cdot\nabla)\mathbf{u}=\nabla\times\mathbf{a},$$

where $\mathbf{u}$ is the velocity and $\mathbf{a}=d\mathbf{u}/dt$ is the acceleration. Consider in the medium a sphere of center P, center of mass $\mathbf{r}_1$, and radius ϵ. Write $\mathbf{L}$ for the total torque on this sphere about its center of mass. Then the author shows that at P

$$\rho\nabla\times\mathbf{a}=\lim_{\epsilon\to 0}\mathbf{L}/\tfrac{1}{5}J,$$

where $J=4\pi\epsilon^5/5$ is the geometrical polar moment of inertia of the sphere about its center. [In the author's paper there are slips or misprints concerning his quantity h, since this eq. (5b) and the words following are not consistent with his eq. (8); the result as quoted agrees with the latter if we take $h=J/3$.]

[It is clear that the result is unaffected if $\mathbf{L}$ is replaced by the torque about any point whose distance from P is $O(\epsilon^\delta)$, $\delta>1$. The author's eq. (12), misnumbered (1), shows that the distance of the center of mass from P is $O(\epsilon^3)$. Hence, in particular, the result could be simplified by taking the torque about P and not mentioning the center of mass at all.]

Multiplying (*) by ρ, the author then interprets the terms on the left as torques. $h\rho d\mathbf{R}/dt$ at P is the torque sufficient for the actual angular acceleration if the spherical element were rigid and of uniform density ρ_P. The author puts

$$(**) \qquad \mathbf{A}=\mathbf{i}R_x\frac{\partial u}{\partial x}+\mathbf{j}R_y\frac{\partial v}{\partial y}+\mathbf{k}R_z\frac{\partial w}{\partial z}.$$

Then $h\rho[(\nabla\cdot\mathbf{u})\mathbf{R}-\mathbf{A}]$ is the torque sufficient to produce a uniform expansion rate outward from the instantaneous axis of rotation equal in magnitude to the actual total expansion at P if the element were of uniform density ρ_P and rotating at constant angular velocity. $h\rho[\mathbf{A}-(\mathbf{R}\cdot\nabla)\mathbf{u}]$ is the torque sufficient to produce the actual rate of shearing in a spherical element of uniform constant density ρ_P. [Note that $\mathbf{A}$, like the concept of shear, is not independent of the choice of co-ordinate system. The author's interpretation is correct only when one of the co-ordinate axes is taken as parallel to the vorticity $\mathbf{R}$.]

The author writes

$$\rho\nabla\times\mathbf{a}=\mathbf{T}_1+\mathbf{T}_2, \quad \mathbf{T}_2=\nabla\times\rho\mathbf{a}, \quad \mathbf{T}_1=-\nabla\rho\times\mathbf{a}.$$

For the case of barotropic flow of a viscous compressible fluid subject to no extraneous force he evaluates $\mathbf{T}_1$ and $\mathbf{T}_2$, then considers a small amplitude alternating velocity field superposed upon a steady flow, then takes time averages. In terms of a perturbation procedure he is then able to interpret the average of certain approximations to $\mathbf{T}_1$ and $\mathbf{T}_2$ as the mean torque due to the alternating stresses of the round field and as the torque due to the viscous stresses of the steady flow, respectively. He then discusses in these terms the approximate streaming equations of Rayleigh, Eckart, and other writers. *C. Truesdell.*

Pearson, J. D. The transient motion of sound waves in tubes. Quart. J. Mech. Appl. Math. 6, 313–335 (1953).

This paper is concerned with the propagation of sound waves of small amplitude along a straight tube with rigid walls, so that the velocity potential satisfies the wave equation and has zero normal derivative on the walls of the tube. Initially the gas is at rest in a state of equilibrium, and a known disturbance is assumed to begin at a particular cross-section of the tube.

The velocity potential can be expanded as a series of eigenfunctions determined by the constant shape of the cross-section of the tube. The expression for the velocity potential is obtained by Heaviside's operational method, but this leads to an integral which is difficult to evaluate. An asymptotic formula for the velocity potential, valid for large values of the time variable, is obtained by the method of steepest descent. The resulting field in the case of harmonic time dependence falls into two parts, a transient motion and the steady state solution; the latter is propagated or attenuated according as the frequency of the forcing field is greater or less than a critical frequency, whereas the transient is propagated with the free space velocity of sound.

The author then goes on to consider the effect of small variations in the cross-section of the tube, on the assumption that this distortion of the tube modifies only slightly the results for the corresponding tube of constant cross-section. The analysis here is rather difficult to follow, but is made clear by a detailed discussion of a tube whose cross-section is nearly rectangular and has a longitudinal harmonic distortion.

E. T. Copson (St. Andrews).

Weston, D. E. The theory of the propagation of plane sound waves in tubes. Proc. Phys. Soc. Sect. B. 66, 695–709 (1953).

The author's abstract reads as follows: "The propagation of plane sound waves in gases in tubes can be divided into three main types, depending on the radius and frequency involved. These types are described as 'narrow' tube, 'wide' tube and 'very wide' tube propagation. The phase velocity, attenuation and cross-section profile of particle velocity, etc. are investigated theoretically, and their inter-relation pointed out. The factors affecting the validity of Kirchhoff's formulae are considered, and the theory is applied to some recent work." *C. J. Bouwkamp* (Eindhoven).

Elasticity, Plasticity

Bordoni, Piero Giorgio. Trasformazioni adiabatiche di ampiezza finita. Ricerca Sci. 23, 1569–1578 (1953).

The author studies some consequences of a formula for the thermodynamic potential of a homogeneous isotropic elastic body which he promises to derive in a forthcoming work. To treat finite adiabatic deformations, he linearizes various expressions with respect to $T-\tau$, where T is the actual temperature and the temperature in the natural state. Thus he obtains a simple formula for $T-\tau$ in terms of the volume change, the strain trace, and various material constants. Hence he finds a condition that a finite deformation can be both adiabatic and isothermal, and he discusses also hydrostatic stress and simple tension. [The analysis rests in part on results in other papers yet to be published, and the reasoning is not always clear to the reviewer. In particular, the reviewer cannot tell whether the results are claimed to hold for general strain energy or only for one of the special forms now frequently considered in the Italian literature.] *C. Truesdell* (Bloomington, Ind.).

Yoshimura, Yoshimaru. On the definition of stress in the finite deformation theory. J. Phys. Soc. Japan 8, 669–673 (1953).

Elaboration of well known results [cf., e.g., the reviewer, J. Rational Mech. Anal. 1, 125–171, 173–300 (1952); 2, 593–616 (1953), §26; these Rev. 13, 794; 15, 178].

C. Truesdell (Bloomington, Ind.).

Langhaar, Henry L. The principle of complementary energy in nonlinear elasticity theory. J. Franklin Inst. 256, 255–264 (1953).

The main object of this paper is to establish that the Legendre transformation is the mathematical device to obtain an expression for the complementary energy of a mechanical system with conservative internal forces. [This fact has been noted by other writers, e.g., G. Hamel, Theoretische Mechanik, Springer, Berlin, 1949, pp. 368–375; these Rev. 11, 548.] As an illustration, a holonomic elastic system with n degrees of freedom and also a non-linear elastic system (small-deflections) are considered; in the latter case a new proof is given for the principle of stationary complementary energy (Castigliano, Engesser).

E. T. Onat (Providence, R. I.).

Gasparini, Ida Cattaneo. Sulla deformazione di un solido elastico omogeneo anisotropo soggetto a stress lineare. Ann. Scuola Norm. Super. Pisa (3) 7, 1–15 (1953).

The author determines the most general displacement field which corresponds to linear stress in the classical linear

theory of anisotropic elasticity for homogeneous bodies. She applies this result to special cases with a view to experimental determination of the 21 elastic coefficients.

C. Truesdell (Bloomington, Ind.).

Džanelidze, G. Yu. General solutions of the equations of the theory of elasticity in arbitrary curvilinear coordinates. Doklady Akad. Nauk SSSR (N.S.) **88**, 423–425 (1953). (Russian)

The author obtains formulas expressing the stress and strain components in arbitrary curvilinear coordinates in terms of three harmonic functions, starting from P. F. Papkovich's [Theory of elasticity, Leningrad, 1939] representation of the displacement vector in terms of harmonic functions. In the special case of orthogonal curvilinear coordinates, the author obtains the formulas given by Sadowsky and Sternberg [J. Appl. Mech. **16**, 149–157 (1949); these Rev. **10**, 760]. The general form of the solution of the equations of elasticity of Galerkin [Doklady Akad. Nauk SSSR **1930**, 353–358] is also obtained as a special case.

J. B. Diaz (College Park, Md.).

Colonnetti, Gustavo. Expression généralisée du théorème de réciprocité. I. C. R. Acad. Sci. Paris **237**, 589–591 (1953).

Colonnetti, Gustavo. Expression généralisée du théorème de réciprocité. II. C. R. Acad. Sci. Paris **237**, 635–637 (1953).

Extensions of Betti's reciprocal theorem are proposed for situations in which stress is not a homogeneous linear function of strain. The precise meaning to be attached to these extensions is not clear. If plastic strains occur then it is necessary to consider strain rates, but this is not done. If there is self-straining then the discussion should involve "barriers" drawn in the body; again this is not done.

H. G. Hopkins (Providence, R. I.).

Payne, L. E. On axially symmetric punch, crack and torsion problems. J. Soc. Indust. Appl. Math. **1**, 53–71 (1953).

This paper deals with familiar problems of elasticity but by methods which reduce considerably the amount of work involved. Solutions of some of these problems have previously been obtained from biharmonic functions satisfying certain prescribed boundary conditions, although Green [Proc. Cambridge Philos. Soc. **45**, 251–257 (1949); these Rev. **10**, 649] has reduced the Boussinesq problem and the crack problem to the determination of a potential function whose first and second normal derivatives are prescribed on each portion of the shear-free plane. The author, using a method suggested by the work of Green, introduces oblate spheroidal and toroidal coordinates and by means of Mehler's integral expansion obtains solutions of the following problems: (i) the spherical punch problem; (ii) the oblate spheroidal punch problem; (iii) the Boussinesq problem with uniform, oblate spheroidal or paraboloidal pressure distributions; (iv) the crack problem with pressure of displacement prescribed. The paper concludes with certain torsion problem analogues. *R. M. Morris* (Cardiff).

Žgenti, V. S. Application of functional analysis in the theory of thin elastic spherical shells. Soobščeniya Akad. Nauk Gruzin. SSR **13**, 257–263 (1952). (Russian)

The author employs the method used in existence theorems in elasticity by K. O. Friedrichs [Ann. of Math. (2) **48**, 441–471 (1947); these Rev. **9**, 255] to establish the existence of the solution, obtainable by Ritz' method, of a certain boundary-value problem in the theory of thin elastic spherical shells. The method consists in showing that a certain operator A, defined in a linear subset M of a Hilbert space which is naturally associated with the boundary-value problem, is positive definite, i.e.

$$(Au, u) \geq \gamma^2 \|u\|^2, \quad \gamma^2 > 0,$$

for u in M, where $(\ ,\ )$ and $\|\ \|$ are the scalar product and the norm in the Hilbert space. [See S. G. Mihlin, Direct methods in mathematical physics, Gostehizdat, Moscow-Leningrad, 1950.] *J. B. Diaz* (College Park, Md.).

Sekiya, Tsuyoshi. On a method of solving the elastic plane stress problems by the aid of integral equation. J. Osaka Inst. Sci. Tech. Part I. **2**, 29–40 (1950).

The author of this paper states that he has reduced the plane stress problem of a multiply-connected region under arbitrary external forces in the plane of the plate to that of the solution of Fredholm's integral equation of the second kind, which he says can be solved numerically. The reviewer finds it impossible to follow the reasoning as there is insufficient explanation of all the symbols used, and there are obviously several printer's errors. *R. M. Morris.*

Tiffen, R. Generalized plane stress problems in infinite elastic strips. Quart. J. Mech. Appl. Math. **6**, 344–369 (1953).

As stated by the author in his summary, "In this paper we obtain solutions in terms of complex potentials to static problems of generalised plane stress for infinite strips of isotropic elastic material. We solve the following types of problems. I. Specified stresses (including isolated forces) along both straight boundaries. II. Specified displacements along both straight boundaries. III. Specified displacements along one straight boundary and specified stresses along the other. IV. Isolated forces in the interior of infinite strips with (a) both boundaries stress free, (b) both boundaries free from displacements, (c) one boundary free from displacements and the other stress-free."

The complex potential functions are obtained in Fourier integral form and to ensure convergence of the stresses and displacements derived from them, at the origin and infinity, care has been taken to modify these functions by the addition of terms which do not affect the stresses or the displacements on the boundaries as the case may be.

R. M. Morris (Cardiff).

Mansfield, E. H. Neutral holes in plane sheet—reinforced holes which are elastically equivalent to the uncut sheet. Quart. J. Mech. Appl. Math. **6**, 370–378 (1953).

A hole made in a plane sheet is called neutral if the stresses acting on its boundary can be held in equilibrium by tangential forces acting on that boundary. Thus the portion of the sheet which has been cut out can be replaced by a reinforcing member which has no bending stiffness. If the state of stress of the sheet is described by the stress-function ϕ, the equation of the boundary of such a hole is of the form $\phi + an + by + c = 0$ where a, b, c are arbitrary constants. A neutral hole can be bounded by arcs of curves each of which is determined by a different set of values of a, b, c provided one applies balancing loads at the function points of adjacent arcs. The expression of the sectional area of the reinforcing member at each point of the boundary and several examples of neutral holes with the corresponding sectional areas of the reinforcing members are given.

C. Arf (Istanbul).

Schultz-Grunow, F. Greensche Funktionen für elastische Platten. Z. Angew. Math. Mech. **33**, 227–237 (1953). (English, French and Russian summaries)

Das Singularitätenverfahren für Rechteckplatten wie es von A. Pucher [Ing.-Arch. **12**, 76–100 (1941); diese Rev. **10**, 86] und W. Koepcke [ibid. **18**, 106–138 (1950); diese Rev. **12**, 142] entwickelt wurde, verfolgt den Zweck, die Greenschen Funktionen $G(\xi_0, \eta_0, \xi, \eta)$ für elastische Platten herzuleiten. Dies Verfahren leidet an einer ungenügenden Erfüllung der Randbedingungen, weil die benutzten biharmonischen Funktionen längs des Randes beliebig oft oszillieren, während dies beim Fundamentalintegral

$$F(\rho) = \rho^2 \ln \rho \quad [\rho^2 = (\xi - \xi_0)^2 + (\eta - \eta_0)^2]$$

nicht der Fall ist. In der vorliegenden Arbeit werden die biharmonischen Funktionen, die dem Fundamentalintegral zu überlagern sind, aus harmonischen Funktionen gewonnen, wodurch funktionentheoretische Hilftsmittel (Schwarz-Christoffel'sche Formel) anwendbar werden, die eine exakte Befriedigung der Randbedingungen in den Ecken und ihrer Umgebung gestatten. Nach Konstruktion der Bipotentialfunktionen wird die Greensche Funktion für die Durchbiegung einer elastischen Platte von quadratischem Grundriss numerisch angegeben. Diese Greensche Funktion wird benutzt, um die Durchbiegung im Mittelpunkt der Platte zu ermitteln, die sich in genauer Übereinstimmung mit den von F. Tölke [ibid. **5**, 187–237 (1934)] und A. Nadai [Forschungsarbeiten Gebiete Ingenieurwesens nos. **170, 171** (1921)] angegebenen Werten ergibt. In der Arbeit sind einige kleinere Druckfehler vorhanden.

R. Gran Olsson (Trondheim).

Capriz, Gianfranco. Sulla applicazione del metodo della trasformata parziale di Laplace ad intervallo di integrazione finito ad un problema di elasticità piana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 401–403 (1953).

Utilisant une idée de Ghizzetti [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 145–187 (1947); ces Rev. **9**, 121], l'auteur applique une transformation de Laplace aux solutions des équations de l'équilibre d'une plaque élastique; il obtient ainsi un système d'équations intégrales élémentaires, dont il discute l'équivalence avec le problème proposé. *J. Deny* (Strasbourg).

Krettner, J. Zur Theorie und Anwendung der schiefen Platte. Ing.-Arch. **21**, 112–118 (1953).

The author presents a detailed tensorial derivation, in oblique coordinates, of the relations which govern small deflections of a thin elastic plate in the form of an oblique parallelogram. No explicit problems are treated.

F. B. Hildebrand (Cambridge, Mass.).

Lovass-Nagy, Viktor. Sur la déformation des plaques minces de forme circulaire et d'une petite courbure constante. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 33–48 (1953). (Hungarian. Russian and French summaries)

The author discusses the bending of a thin, slightly spherically curved, circular plate under the influence of hydrostatic pressure. Both clamped and freely supported plates are considered, and the displacement of the center and normal stresses in the plate are computed.

A. Erdélyi (Pasadena, Calif.).

Lovass-Nagy, Viktor. Sur la flexion d'une plaque circulaire comprimée dans son propre plan. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 23–32 (1953). (Hungarian. Russian and French summaries)

The author discusses the buckling of circular plates under the influence of a uniform normal pressure in the plane of the (undeformed) plate. Both clamped and freely supported plates are discussed and the method is modelled on the Kármán-Biot method for the corresponding problems regarding beams. *A. Erdélyi* (Pasadena, Calif.).

Mitra, D. N. Flexure of an isotropic elastic cylinder whose cross-section is bounded by two closed curves. Bull. Calcutta Math. Soc. **44**, 143–151 (1952).

The mapping function for the ring space between the two closed curves in the z-plane being $z = \sum_{-\infty}^{\infty} a_n \sigma^n$, it transforms the region conformally into a region bounded by two concentric circles in the σ-plane. In that this mapping function contains any number of negative powers of σ, it is an extension of the results already found by the reviewer [Proc. London Math. Soc. (2) **51**, 424–439 (1949); these Rev. **10**, 650] who used the transformation $z = \sigma^{-1} \sum_{n=1}^{\infty} a_n \sigma^n$, although the author does not give any examples of regions not covered by the latter transformation. The required complex functions are found in compact forms, as well as the position of the centre of flexure. The results are applied to the ring space between two non-concentric circles and they are found to agree with those of other authors. *R. M. Morris.*

Karunes, B. On the distribution of stress round the edge of a hole in a deep beam under a uniform bending moment. Indian J. Phys. **27**, 373–378 (1953).

Fralich, Robert W., Mayers, J., and Reissner, Eric. Behavior in pure bending of a long monocoque beam of circular-arc cross section. NACA Tech. Note no. **2875**, 33 pp. (1953).

An analysis is made of the behaviour under a loading of pure bending of a thin, infinitely long, pure-monocoque beam having a constant, doubly symmetric, circular-arc cross-section. The flattening of the cross-section cannot be neglected and causes a non-linear relation between bending moment and beam curvature, which leads ultimately to instability. If no interaction is assumed between local buckling of the compression cover and flattening instability, local buckling should precede flattening instability of the beam when the radius-thickness ratio of the covers is greater than about 720 and flattening instability should precede local buckling when this ratio is smaller than about 720. *W. H. Muller* (Amsterdam).

Broglio, Luigi. Questioni analitiche inerenti ai problemi di stabilità dell'equilibrio elastico. Pont. Acad. Sci. Acta **15**, 15–24 (1951). (Latin summary)

The solution of a problem in elastic stability involves consideration of second-order terms in the strains. Here expressions, appropriate to various circumstances, are given for the components of finite strain in an elastic body.

H. G. Hopkins (Providence, R. I.).

Svešnikov, A. G. Uniqueness of solution of exterior problems of the theory of elastic vibrations. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 443–454 (1953). (Russian)

The unique determination of the solution in the exterior boundary-value problems in the theory of vibrations requires the imposition of certain "radiation conditions" at

infinity [see, e.g., V. D. Kupradze, Boundary value problems of the theory of vibrations and integral equations, Gostehizdat, Moscow-Leningrad, 1950; I. N. Vekua, Doklady Akad. Nauk SSSR (N.S.) **80**, 341–343 (1951); these Rev. **14**, 336]. The author points out that these conditions depend upon the particular domain in question, in that there are simple domains for which there exists no solution to the problem satisfying these "uniqueness producing" conditions, and sets himself the problem of finding a more general (i.e., allowing existence of a solution) method for uniquely determining the solution of the exterior boundary-value problems in elasticity. His method, which he designates by the phrase "principle of limiting absorption", consists in seeking the solution of the equation (Δ is the Laplacian)

$$\Delta u+k^2u=-f \quad (k \text{ real})$$

as the limit of solutions of the equation

$$\Delta u+k_1^2u=-f \quad (k_1=k+i\epsilon;\ \epsilon>0)$$

which are bounded at infinity. In an earlier paper [Svešnikov, ibid. **73**, 917–920 (1950); these Rev. **12**, 233], this principle of limiting conductivity was applied for the unique determination of solutions of the scalar wave equation. In the present paper, the more difficult vector wave equation occurring in steady elastic vibrations is treated.

J. B. Diaz (College Park, Md.).

Melyahoveckiĭ, A. S. Oscillation properties of vibrations of a compressed beam. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 461–464 (1953). (Russian)

The author considers small transverse oscillations of a compressed beam with arbitrary piece-wise continuous axial loading $Pf(x)$, one end being fixed and the other free, hinged or fixed. For $P<P_1$, the natural modes have the Sturmian oscillation properties. The proof is based on a study of the Green's functions. *F. V. Atkinson* (Ibadan).

Caligo, Domenico. Nuovi complementi analitici e numerici allo studio delle aste vibranti. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 36–42, 223–230 (1953).

This note outlines the application of certain well known methods to the analysis of rotating shafts.

F. B. Hildebrand (Cambridge, Mass.).

Drobot, S. On torsional vibrations of a shaft. Arch. Méc. Appl., Gdańsk **3**, 127–146 (1951). (Russian. Polish summary)

This paper deals with the calculation of the natural torsional frequencies of a crank-shaft, idealized to a massless elastic shaft supporting heavy discs. The well known theory is expounded again. First the homogeneous shaft of the nth order is dealt with, i.e., $n+1$ identical discs connected by n identical shafts. Then these results are applied to the case in which two or three unequal discs are added. The article closes with a numerical example and a nomogram for finding the frequencies of a homogeneous shaft of the 4th order with two additional unequal masses. *W. H. Muller.*

Haskell, Norman A. Asymptotic approximation for the elastic normal modes in a stratified solid medium. Air Force Cambridge Research Center, Cambridge, Mass., Geophysical Research Papers No. **22**, 24 pp. (1953).

Wave propagation in a semi-infinite solid medium is considered, in which the density ρ and Lamé constants λ and μ are assumed to be slowly varying functions of the depth z. If the displacements are expressed in terms of functions which satisfy the conditions for the case of constant ρ, λ, μ, the equations for factors depending on z are approximately valid when ρ, λ, μ are variable provided their relative variations within a wavelength are very small. These equations are of the form $Z''+(\omega^2/v^2-k^2)Z=0$ where $v=\alpha=\alpha(z)$ or $\beta=\beta(z)$, ω is the frequency and k the wave number. Using two linearly independent solutions of each of these equations the arbitrary constants involved are determined from boundary conditions and as usual a point source solution is represented as a sum of normal modes. If this approximate theory is applied to a medium made up of two or three homogeneous layers it yields a fairly good representation of the phase velocity curves for the normal modes of order higher than the fifth in a certain range for each mode. *W. S. Jardetzky* (New York, N. Y.).

Jobert, G., et Jobert, N. Application du principe de Rayleigh à la dispersion d'ondes superficielles. Ann. Géophysique **9**, 250–255 (1953).

The Jeffreys method [Monthly Not. Roy. Astr. Soc. Geophys. Suppl. **3**, 253–261 (1935)] is applied to a heterogeneous half-space in which the density and elastic properties are represented by continuous functions of the depth z. The approximate values of integrals the ratio of which determines the phase velocity of Rayleigh waves are computed for a wave number k differing but little from zero (terms of the first and second order) and under special assumptions concerning the medium (the rigidity μ and ρ/μ can be namely expanded in power series with respect to the variable $u=e^{-mz}$). *W. S. Jardetzky* (New York, N. Y.).

***Hoffman, Oscar, and Sachs, George. Introduction to the theory of plasticity for engineers.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. xiv+276 pp. $6.50.

This book on the theory of plasticity is designed to be used as a text by advanced engineering students and also as a reference book by practising engineers. Advanced mathematics is not assumed to be familiar to the reader; thus the necessary parts of elementary tensor analysis are developed simply in early sections on stress and strain. The book has four parts. Part I discusses the concepts of stress and strain, theories of strength, and stress-strain relationships. Parts II and III give the solution of a variety of problems in the plastic flow of ideally-plastic materials and of strain-hardening materials, respectively. Part IV comprises nearly one half of the book, and is devoted to the theory of the various important metal-forming processes, such as drawing, extrusion and rolling. The book is well-written and amply illustrated. It will fill a gap in the literature on the theory of plasticity and its technological applications. The presentation will appeal to the type of reader that the authors have in mind, and the book should also serve a useful purpose in stimulating the interest of research workers in the development of this interesting and important subject. *H. G. Hopkins* (Providence, R. I.).

***Geiringer, Hilda. Some recent results in the theory of an ideal plastic body.** Advances in Applied Mechanics, vol. 3, pp. 197–294. Academic Press Inc., New York, N. Y., 1953. $9.00.

A mathematical account is given of the two-dimensional, perfectly-plastic rigid theory of deformation in isotropic continua. The paper was written in the fall of 1951. The aim is to give a coherent presentation, and to recognize

fundamental difficulties where these occur. The treatment is extremely detailed, and, although it is not intended to be complete in respect to solutions to particular problems, the paper will be of considerable value to those with interests in this part of plasticity theory.

There are three main parts. In Part I, various plane problems are defined, and their governing equations are set up from the general three-dimensional equations of plasticity. In particular, the equations for plane stress and plane strain are given, and the plane stress problem involving changes in plate thickness is also discussed. Part II discusses aspects of the general plane problem that includes the plane strain and plane stress problems. Recent work strongly favors the coincidence of the yield function and the plastic potential, and here this is usually, but not always, taken to be the case. Linearization, characteristics and simple waves are the main topics treated. This part of the paper is largely based upon the author's own work. Part III gives some discussion of complete solutions, and refers principally to work by W. Prager and E. H. Lee. The importance of solving a problem throughout the plastic and rigid regions is emphasized. *H. G. Hopkins* (Providence, R. I.).

Neuber, H. Bruchflächentheorie für das ebene anisotrope Plastizitätsproblem. Z. Angew. Math. Mech. **33**, 293–299 (1953).

The paper is concerned with plane plastic stress fields in anisotropic media. The discussion is based on the following tacit assumptions: the medium is homogeneous and orthotropic, the plane of stress coinciding with one of the planes of orthotropy. The coordinate directions x, y in this plane are taken along the axes of orthotropy, and the normal and shearing stresses with respect to these axes are denoted by σ_x, σ_y, and τ_{xy}. The yield condition is represented by a surface in a stress space with the rectangular coordinates $\sigma_x+\sigma_y$, $\sigma_x-\sigma_y$, and $2\tau_{xy}$. For an isotropic medium this yield surface is rotationally symmetric with respect to the $\sigma_x+\sigma_y$ axis; for an anisotropic medium the yield surface is symmetric with respect to the plane $\sigma_x-\sigma_y=0$. The characteristic directions in the (x, y)-plane (physical plane) are obtained from the following question: can the equations of equilibrium be combined in such a manner that the resulting equation involves only derivatives in a single direction? There are two such directions at each point of the stress field, and the characteristics indicate these directions. The images of the characteristics in the physical plane are found to be helical lines that form angles of 45° with the axis of $\sigma_x+\sigma_y$. The geometrical relations between the orthogonal projections of these lines on the plane $\sigma_x+\sigma_y=0$ and the characteristics in the physical plane are investigated. It is shown that the characteristics indicate the orientations of surface elements for which the tractions reach limiting values. *W. Prager* (Providence, R. I.).

Jung, H. Der rotationssymmetrische elastisch-plastische Körper. Österreich. Ing.-Arch. **7**, 168–180 (1953).

The author proposes a stress-strain law for large, axially symmetric deformations of a strain-hardening elastic-plastic material. (This law is of the "finite" (as opposed to "incremental") type; the results obtained from it must therefore be questioned even in the case of axial symmetry.) First and second order approximations are discussed, and the following examples are treated: tension of a solid circular cylinder and tension of a circular tube under interior pressure. *W. Prager* (Providence, R. I.).

Hopkins, H. G., and Prager, W. The load carrying capacities of circular plates. J. Mech. Phys. Solids **2**, 1–13 (1953).

The paper is concerned with the determination of the load carrying capacity of thin circular plates made of a plastic-rigid material that obeys Tresca's yield condition and the associated flow rule. The discussion is restricted to rotationally symmetric types of loading and edge support, and the exact solutions to the cases investigated are found. The authors observe that the simplification achieved by the use of Tresca's yield condition is considerable. In an earlier paper [Pell and Prager, Proc. First U. S. Nat. Congress Appl. Mech., Chicago, 1951, Amer. Soc. Mech. Engrs., New York, 1952, pp. 547–550; these Rev. **14**, 930] the simply supported circular plate under uniformly distributed loading was discussed under the assumption that the plate material obeys the yield condition and flow rule of v. Mises. The mathematical work was so much more complex in this case that an exact evaluation of the load carrying capacity proved too difficult. The best result that could then be achieved bounded this capacity to within ±5.3 per cent. It is interesting to note that this approximate value of the load carrying capacity does not differ greatly from the exact value obtained for the yield condition of Tresca and the associated flow rule.

E. T. Onat (Providence, R. I.).

Lovass-Nagy, Viktor. Sur les états de tension plastiques et élastiques dans les tuyaux à paroi épaisse. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 49–80 (1953). (Hungarian. Russian and French summaries)

The state of stress and deformation is calculated in an indefinitely long thick-walled tube subjected to uniformly distributed radial tractions on the internal and external surfaces. It is assumed that a plane state of stress prevails in the tube. The radius of the cylindrical surface is calculated which divides the wall of the tube into an elastic and a plastic region. Both the Saint-Venant-Tresca and the von Mises criteria of plasticity are used in the analysis. In a fully worked out numerical example the difference between the radii of the limiting surface according to the two criteria is found to be ten percent. The Saint-Venant-Tresca condition is recommended in practical work because it is simpler to apply. Nomograms are presented to facilitate the computation of the radius of the limiting surface. The paper is clearly written and is easy to read. *N. J. Hoff.*

Manacorda, Tristano. Sul legame sforzi-deformazione nelle trasformazioni finite di un mezzo continuo isotropo. Rivista Mat. Univ. Parma **4**, 31–42 (1953).

The author finds a geometric motivation for stress-strain relations which seem to the reviewer to be identical with those given by Reiner [Amer. J. Math. **70**, 433–446 (1948); these Rev. **10**, 81] as the general form for isotropic bodies, except that the author adds a perfectly arbitrary hydrostatic tension. He shows how several other incremental laws are included as special cases. He claims that the v. Mises plasticity equations are similarly included, but the analysis here confuses rates with displacements and employs a limiting process.

C. Truesdell (Bloomington, Ind.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Irons, Eric. J. Some generalizations in geometrical optics derived by a convergence method. Amer. J. Phys. **21**, 590–602 (1953).

Poincelot, Paul. Extension du principe de Fermat au temps de propagation de groupe. C. R. Acad. Sci. Paris **237**, 382–384 (1953).

In a dispersive medium, the rays for frequency ω satisfying $\delta\int ds/u=0$ where $u(x, y, z, \omega)$ is the phase velocity, the author assumes the existence of two adjacent rays C_1, C_2, both emanating from a point A and meeting again at a point B; C_1 corresponds to frequency ω and C_2 to frequency $\omega+\delta\omega$. He deduces by a very simple argument that the quantity

$$(*) \qquad \left(\int_{C_1} ds/g - \int_{C_2} ds/g\right)\delta\omega$$

is small of at least the second order, g being the group velocity. He concludes that the variation of the time of propagation of the group is zero when one passes from the real trajectory C_1 to the real trajectory C_2: $\delta\int_A^B ds/g=0$. [This conclusion appears incorrect to the reviewer; from the fact that the quantity (*) is small of at least the second order, one can deduce only that $\delta\int ds/g$ is small of at least the first order, a trivial result since it is true for any variation of any integral.] *J. L. Synge* (Dublin).

Meixner, J. Theorie der Beugung elektromagnetischer Wellen an der vollkommen leitenden Kreisscheibe und verwandte Probleme. Ann. Physik (6) **12**, 227–236 (1953).

The diffraction through a circular aperture in a perfectly conducting plane screen or at the complementary circular disc, for incident spherical electric or magnetic dipole waves, is presented in terms of oblate spheroidal wave functions. There is no restriction as to the location and direction of the dipole source. Certain symmetrical diffraction problems (where the electric field is solenoidal) for oblate spheroids of non-vanishing thickness are also considered.

C. J. Bouwkamp (Eindhoven).

Bekefi, G. Diffraction of electromagnetic waves by an aperture in a large screen. J. Appl. Phys. **24**, 1123–1130 (1953).

An approximate solution for the diffraction of an electromagnetic wave by an aperture in a plane conducting screen is obtained from a one-component Hertzian vector [cf. F. Möglich, Ann. Physik (4) **83**, 609–734 (1927)]. The results are applied to calculating the field of apertures sufficiently large compared to the incident wavelength. An infinitesimal-wavelength approximation leads to a Kirchhoff-like formula. The case of diffraction of a plane wave at normal incidence by a circular aperture is treated in detail, and the results of theory are compared with published experimental findings.

C. J. Bouwkamp (Eindhoven).

Harrington, Roger F. Current element near the edge of a conducting half-plane. J. Appl. Phys. **24**, 547–550 (1953).

The classical problem of diffraction by a half-plane is treated by Wiener-Hopf techniques for the case of line-source excitation. [For the somewhat simpler case of plane-wave excitation cf. Copson, Quart. J. Math., Oxford Ser. **17**, 19–34 (1946); these Rev. **8**, 29.] *S. Karp.*

Agostinelli, Cataldo. Sulla propagazione di onde elettromagnetiche guidate entro tubi cilindrici. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 121–147 (1952).

The author considers in detail the propagation of electric and magnetic type waves in conducting cylinders filled with heterogeneous dielectric. The dielectric constant is considered to be invariant along lines parallel to the axis, and in special cases, to have values which are axially symmetric. Harmonic variation of time and propagation of the $e^{-i\beta z}$ form is assumed and the following is demonstrated. 1. There exists a lower limit of the phase velocity which depends on the minimum value of ϵ. 2. For electric type waves the tube boundary (not necessarily circular) coincides with a line of constant ϵ. In fact, under the restriction of the invariance of the dielectric constant with z, the electric type is only possible in a tube of circular or coaxial section. 3. Waves of the magnetic type require a conductor of circular, coaxial, or a segment of either, form as well as axial symmetry of stratification. The problem reduces to an integral equation with a continuous spectrum. This is solved under the restriction of a certain parabolic variation of dielectric constant with radius and propagation at the minimum phase velocity.

W. K. Saunders (Washington, D. C.).

Zeuli, Tino. Sulla propagazione di onde elettromagnetiche critiche entro tubi cilindrici circolari e rettangolari con dielettrico eterogeneo. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 297–328 (1952).

The author continues the work described in the paper reviewed above with particular attention to waves which propagate at the minimum phase velocity (a function of the minimum dielectric constant). In this the phase velocity is taken as known and the author works backward to the possible harmonic excitations which will produce it. Both linear and parabolic variation of dielectric constant is considered. (The reviewer fails to see the physical significance of the minimum value of dielectric constant, especially since this might occur on a set of measure zero.)

W. K. Saunders (Washington, D. C.).

Moon, Parry, and Spencer, Domina Eberle. TEM waves in cylindrical systems. J. Franklin Inst. **256**, 325–336 (1953).

***Deschamps, G. A. Geometric viewpoints in the representation of waveguides and waveguide junctions.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 277–295. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

Lüst, R. Magneto-hydrodynamische Stosswellen in einem Plasma unendlicher Leitfähigkeit. Z. Naturforschung **8a**, 277–284 (1953).

The equations of hydromagnetic shocks as developed by de Hoffmann and Teller [Physical Rev. (2) **80**, 692–703 (1950); these Rev. **12**, 769] are examined in some detail and the manner in which these equations pass into the usual equations of Rankine-Hugoniot (when there is no magnetic field) is examined. The limit to which the shock equations tend as the amplitude of the discontinuity becomes small

is also considered. [The discussion is less extensive and less complete than the one by Helfer, Astrophys. J. 117, 177–199 (1953); these Rev. 14, 804.] *S. Chandrasekhar.*

Dungey, J. W. A family of solutions of the magneto-hydrostatic problem in a conducting atmosphere in a gravitational field. Monthly Not. Roy. Astr. Soc. 113, 180–187 (1953).

The equations governing the hydrostatic equilibrium of a system in which there is a prevalent magnetic field, **H**, are

$$\mathbf{j}\times\mathbf{H}+\rho\mathbf{g}-\text{grad}\,p=0 \quad \text{and} \quad \mathbf{j}=\text{curl}\,\mathbf{H}/4\pi,$$

where **j** denotes the current density, **g** the accelerations due to gravity (directed along the vertical, z), p the pressure and ρ the density. In two dimensions the magnetic field can be derived from a potential A (such that $H_z=-\partial A/\partial z$ and $H_x=\partial A/\partial x$) and the equations become

$$(1)\qquad j=j_y=-\frac{1}{4\pi}\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial z^2}\right), \quad j_z=j_x=0,$$

$$(2)\qquad j\frac{\partial A}{\partial z}=e^{-\alpha z}\frac{\partial q}{\partial z} \quad \text{and} \quad j\frac{\partial A}{\partial x}=e^{-\alpha z}\frac{\partial q}{\partial x},$$

where $q=e^{\alpha z}p$. From equations (2) it follows that q is a function of A and that $j=e^{-\alpha z}dq/dA$. With the further assumption "for any magnetic field having these lines of force, a distribution of the gas exists which will be in equilibrium with this field", the author discusses some special cases of the foregoing solution. *S. Chandrasekhar.*

Higgins, Thomas James, and Reitan, Daniel Kinseth. Calculation of the capacitance of a circular annulus by the method of subareas. Trans. Amer. Inst. Elec. Engrs. 70, 926–933 (1951).

Baudoux, Pierre. Sur le potentiel vecteur dû aux courants de surface. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 39, 636–639 (1953).

Miyatake, Osamu. Coulomb energy of a uniformly charged liquid-drop. II. J. Inst. Polytech. Osaka City Univ. Ser. B. Physics 3, 67–70 (1952).

[For part I see Miyatake and Watanabe, same J. 2, 15–22 (1951); these Rev. 14, 750.] An approximation to the electrostatic energy of a uniformly charged solid of revolution is given in a series of stages involving solids of equal volume but slightly varying shapes, beginning with a solid whose electrostatic energy is known. *C. Strachan* (Aberdeen).

Ashour, A. A. Electromagnetic induction in shells and disks having special distributions of conductivity. Proc. Math. Phys. Soc. Egypt 4 (1952), no. 4, 9–16 (1953).

The differential equation in general orthogonal curvilinear coordinates determining the eddy current distribution in thin shells simplifies greatly if one assumes the conductivity as an appropriate function of one of the coordinates. The application to the oblate spheroidal shell is shown for both simple harmonic time variation and the Heaviside unit step. As a special case, the induced current distribution is found in a symmetrically conducting circular disk. *E. Weber.*

Freud, Géza. Sur le calcul du champ magnétique d'une conduite électrique parallèle. I. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 1 (1952), 377–387 (1953). (Hungarian. Russian and French summaries)

The author computes the magnetic field generated by electric currents of equal magnitude and opposite direction flowing in two parallel cylinders. The frequency is assumed to be small, the field is assumed quasi-stationary, and the changes of current distribution in each cylinder which are due to the presence of the other cylinder are taken into account only as corrections. The results are expressed as infinite series which converge at least as well as geometric progressions; if the radii of the cylinder are small in comparison with the mutual distance, the convergence is very rapid. *A. Erdélyi* (Pasadena, Calif.).

***Foster, R. M. Topologic and algebraic considerations in network synthesis.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 8–18. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

Expository paper.

Bulgakov, B. V. Questions of the synthesis of correcting circuits. I, II. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1952, 21–40, 699–723 (1952). (Russian)

For a given rational function analytic in the right half-plane and on the imaginary axis which is positive-real in the sense of Brune [J. Math. Physics 10, 191–236 (1931)], a method is given for realizing a four-pole passive lumped-constant network without mutual inductances for which this function is the transmission function up to a multiplicative real constant. The procedure depends on factoring the rational function into fractions whose numerators and denominators are of degree not greater than two. Detailed formulas for the determination of symmetric bridge four-pole networks for all fractions of this type are given. The network formed by the cascade connection of the networks representing the factors of the given transmission function is the desired network. *C. Saltzer* (Cleveland, Ohio).

Mason, Samuel J. Feedback theory. I. Some properties of signal flow graphs. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. No. 153, 25 pp. (1953).

Graphs with fixed, oriented branches are considered where paths are defined only for certain configurations. The oriented branches denote functional relations for the node weights; a reduction process which yields graphs with a minimum number of relations, is described. Also, invariants of the oriented structures are defined. An interpretation in terms of programming is given, and the application to systems for which the functions involved are linear is discussed in detail and applied to multiple loop servo-mechanisms, electronic circuits, and transmission lines. *C. Saltzer.*

Kautz, William H. Network synthesis for specified transient response. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. No. 209, iii+177 pp. (1952).

A filter network N transforms a function of the time $f_i(t)$ into another $f(t)$. The transient response $f(t)$ may be approximated as a polynomial $f^*(t)$ whose terms are damped sinusoids. Then if f_i is an impulse, there are known methods of finding a finite lumped network N^* which transforms f_i into f^*. This paper treats the problem of determining N^* with consideration of network economy, simplicity in calculation, and error control. The problem of a suitable choice of the damped sinusoids is analyzed. Attention is given to additional constraints which are imposed by practical limitations. The Laplace transforms of the above functions are denoted by $F_i(s)$, $F(s)$, and $F^*(s)$. Then F/F_i determines

the frequency response of the network N. Consideration is given to the question of the approximation of F^* to F. A method is given employing approximation, both in the time domain and in the frequency domain. The mathematical techniques employed include orthogonal polynomials, saddle-point methods, continued fractions, and approximation theory. Numerous examples, tables, and graphs are given concerning the synthesis methods developed.

R. J. Duffin (Pittsburgh, Pa.).

Ozaki, Hiroshi. Synthesis of RC-3-terminal network without ideal transformer. Tech. Rep. Osaka Univ. **3**, nos. 55–73, 57–77 (1953).

Let N be a four-terminal RC network with impedance matrix elements z_{11}, z_{12}, and z_{22}. Suppose that none of the zeros of z_{12} are to the right of the imaginary axis in the complex-frequency plane and that $(z_{22}-z_{12})/(z_{11}-z_{12})=n$ where n is a positive number. It is shown that such a network N may be synthesized as a three-terminal RC network. The procedure is given for carrying out the synthesis. A simple transformation yields a similar theorem for LC networks. *R. J. Duffin* (Pittsburgh, Pa.).

***Cherry, E. C. Application of the electrolytic tank techniques to network synthesis.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 140–160. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

***Trautman, DeForest L. The application of conformal mapping to the synthesis of bandpass networks.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 179–192. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

Ku, Y. H. Transient analysis of rotating machines and stationary networks by means of rotating reference frames. Trans. Amer. Inst. Elec. Engrs. **70**, 943–957 (1951).

van der Pol, Balth. Radio technology and the theory of numbers. J. Franklin Inst. **255**, 475–495; addendum and correction **256**, 265 (1953).

Quantum Mechanics

Wiener, Norbert, and Siegel, Armand. A new form for the statistical postulate of quantum mechanics. Physical Rev. (2) **91**, 1551–1560 (1953).

If $U(t)$ is the unitary operator corresponding to translation in time by the amount t, for a quantum-mechanical system with time-independent Hamiltonian, it is in general not true that the action of $U(t)$ on state vectors ψ arises from a group of measure-preserving transformations on the space on which the ψ's are defined. In fact the action need not even be unitarily equivalent to that arising from such a group. However, there exists a probability space Ω with the property that the states ψ are representable as (certain of the) elements of $L_2(\Omega)$ (i.e. the space of square-integrable functions on Ω), and the action of the $U(t)$ on the ψ's is then representable by means of a one-parameter group of probability-preserving point transformations on Ω. The space Ω can conveniently be taken to be the continuous functions on the space on which the ψ's are defined, with Wiener (Brownian notion) measure on it. This result is implicit in "Fourier transforms in the complex domain" [Amer. Math. Soc. Colloq. Publ., v. 8, New York, 1934], by Paley and Wiener, and is here given a detailed but heuristic exposition.

Now if T is an observable and ψ a state vector, there exists a function F which assigns to each point of Ω a point in the spectrum of T, and has the property that if A is any (Borel) set, then the probability of T having a value in A, in the state ψ, in the conventional quantum-mechanical sense, is the Wiener measure of the set of points ω in Ω for which $F(\omega)$ is in A. This is mathematically equivalent to the assertion that given a distribution function on the line, there exists a random variable on Wiener space having it as its distribution function. Actually, the space $L_2(\Omega)$ is identifiable in a natural way with the space of state vectors for a free field of bosons with states ψ, and the notion of such a field is thus always unitarily equivalent to a motion induced by a group of probability-preserving point transformations.

I. E. Segal (New York, N. Y.).

Thermodynamics, Statistical Mechanics

Genzel, L. Der Anteil der Wärmestrahlung bei Wärmeleitungsvorgängen. Z. Physik **135**, 177–195 (1953).

Consideration is given to one-dimensional energy flow and the associated temperature gradient existing in conjunction with a thermal radiation field in an absorbing medium. The basic formulation assumes a given temperature distribution, a frequency-dependent absorption coefficient, a constant material-dependent index of refraction, and radiation in accord with Planck's law. General results are stated for infinite, semi-infinite, and finite one-dimensional media. Simplifications for the cases of piecewise constant absorption coefficient and uniform or 'grey' absorption indicate consistency with established relations. The combined action with conventional conduction is developed for both the general and simplified cases. For a constant energy flow in a finite medium with black-body surfaces, the temperature distribution with and without conventional conduction is obtained as a function of a frequency-independent absorption coefficient. *N. A. Hall.*

Onsager, L., and Machlup, S. Fluctuations and irreversible processes. Physical Rev. (2) **91**, 1505–1512 (1953).

The time variation of the fluctuations in a thermodynamical system are studied under the following assumptions: (i) on the average the fluctuations obey the phenomenological equations of irreversible processes, (ii) the fluctuating thermodynamic variables have a Gaussian probability distribution. The probability for a fluctuation to have given values at a finite number of given times is calculated and expressed in terms of a conditional minimum for the total dissipation, if the latter is suitably written in terms of the thermodynamic variables and their time derivative. The treatment is restricted to systems, the states of which can be completely specified by means of variables invariant under time inversion (see the following review).

L. Van Hove (Princeton, N. J.).

Machlup, S., and Onsager, L. Fluctuations and irreversible process. II. Systems with kinetic energy. Physical Rev. (2) **91**, 1512–1515 (1953).

The discussion of the preceding paper is extended to systems with thermodynamical variables, some of which change sign under time inversion. *L. Van Hove.*

March, N. H. The virial theorem for Dirac's equation. Physical Rev. (2) **92**, 481–482 (1953).

Berlin, T. H., Witten, L., and Gersch, H. A. The imperfect gas. Physical Rev. (2) **92**, 189–201 (1953).

The partition function of a classical system of interacting particles is calculated for an approximate model, similar to the spherical model of ferromagnetism previously proposed by Berlin and Kac [Physical Rev. (2) **86**, 821–835 (1952); these Rev. **14**, 232]. The system is subdivided into cells and the interaction energy between pairs of particles is approximated by its value at the centers of the cells. The occupation numbers of the cells are treated as continuous variables, for which negative values are admitted but made less probable than positive values by imposition of a quadratic condition. Under these approximations the partition function can be calculated explicitly. It is found that below a critical temperature the pressure, considered as function of volume per particle, is represented in three regions by three different analytic functions. The middle region is likely to correspond to the physical condensation region, although the model does not give a constant pressure. At sufficiently low temperatures the model suffers from thermodynamic instability. The correlation between numbers of particles in different cells is calculated. *L. Van Hove.*

Baijal, J. S., and Chandra, R. Degenerate gas and the motion of a particle in a harmonic field. Proc. Nat. Inst. Sci. India **19**, 423–426 (1953).

This paper gives an evaluation of some properties of a completely degenerate ideal Fermi-Dirac gas in an external potential field, $V(x)=\infty$ for $x<0$, $V(x)=\frac{1}{2}\alpha x^2$ for $x>0$. A gas subject to an external field composed of the sum of such fields acting along mutually perpendicular spatial directions is also considered. The notation and style follow that of Kothari and Auluck [same Proc. **8**, 157–164 (1942)] who made a similar analysis with $V(x)$ proportional to x for $x>0$. *G. Newell* (Providence, R. I.).

de Groot, S. R., Hooyman, G. J., and ten Seldam, C. A. On the Bose-Einstein condensation. Proc. Roy. Soc. London. Ser. A. **203**, 266–286 (1950).

This is a discussion of one-, two- and three-dimensional perfect Einstein-Bose gases. The well known results of F. London [Physical Rev. (2) **54**, 947–954 (1938)] on the condensation of a three-dimensional gas are derived. It is shown that one- and two-dimensional gases do not have a sharp condensation point. Of course, the condensation is a limit result as the number of particles and the container volume become infinite at a fixed density.

Besides considering the behavior of particles in a box with "rigid" walls, the authors examine the case of soft "gaussian" walls. This corresponds to a system of harmonic oscillators with Einstein-Bose statistics. As in the hard-wall case, a condensation occurs in the three-dimensional case but not in the two. While the heat capacity is a continuous function of the temperature with a discontinuous first derivative in the hard-wall system, it is a discontinuous function in the gaussian wall case. *E. W. Montroll.*

Scoins, H. I. A note on the linearized integral equation of Green. Philos. Mag. (7) **43**, 806–808 (1952).

Green's treatment of the linearized integral equation for liquids proposed by Born and himself [H. S. Green, Proc. Roy. Soc. London. Ser. A. **189**, 103–117 (1947); these Rev. **9**, 402] is shown to contain an error for the case of a kernel with real poles. The correct solution gives rise to a slowly decreasing pair distribution function and to infinite X-ray scattering for a certain angle. The author suggests that the latter might be small and that the infinite scattering might thus have escaped observation. No justification is given for this highly unconventional view. *L. Van Hove.*

Zimm, B. H., Stockmayer, W. H., and Fixman, M. Excluded volume in polymer chains. J. Chem. Phys. **21**, 1716–1723 (1953).

The problem which is described by the title of this paper is one that has been receiving considerable attention recently. In chemical language, it deals with questions regarding the statistical distribution of atoms in very large molecules (polymers) assuming that a molecule behaves like a chain of atoms, each atom bound tightly at a fixed distance from the preceding element of the chain but repelled by all other elements so that the chain is hindered from twisting back upon itself. Mathematically the problem corresponds to a random walk problem in which each new step (atom) of the walk prefers to go in such a direction that it does not go too near the end of any previous step of the sequence. Thus, one has a Markov chain of infinite order.

The authors give a mathematical formulation of a rather general system of this type. They show that although the mean square radius of a chain of N (finite) atoms can not be easily evaluated explicitly, it can be related at least approximately to the second virial coefficient for the osmotic pressure of a solution of such chain molecules. This relation is compared with experimental data and shown to be in reasonable agreement. On the basis of this data they conclude that the effect of the forces of repulsion (excluded volume effect) between non-bonded atoms is not negligible and they give a few qualitative arguments that the mean square radius of such chains probably increases faster than order N asymptotically (unlike the free random walk or random walk with only short range correlations). The purely mathematical questions regarding the asymptotic behavior of such chains, however, still appear a long way from giving a satisfactory answer. *G. Newell.*

Syozi, Itiro. Statistical mechanics of two dimensional lattices. Annual Rep. Sci. Works Fac. Sci. Osaka Univ. **1** (1952), 1–18 (1953).

This article discusses the various two-dimensional lattices for which solutions are known and the methods by which one passes from one problem to another. These latter include the dual transformation and the "star triangle" transformation. A new treatment by Kano, Naya and Utiyama, hitherto unpublished, of the rectangular lattice is given. Utiyama also succeeded in treating certain additional lattices which are described. The relation of the solution with thermodynamic and magnetic properties are given. There is also a brief discussion of the work of Kac and Ward and of Berlin and Montroll. *F. J. Murray.*

Murray, F. J. The Curie point in the three-dimensional order-disorder problem. Ann. of Math. (2) **55**, 250–279 (1952).

The paper describes the algebraic foundation for a half analytical, half computational treatment of the three-dimensional order-disorder problem. The approach starts from the low temperature region and makes use of the work of van der Waerden [Z. Physik **118**, 473–488 (1941)] on the existence of long range order in that region. The method is based on assumptions which are not proved but made plausible. *L. Van Hove* (Princeton, N. J.).